Lecture 3: Dot product (12.3) and lines and planes in \( \mathbb{R}^3 \) (12.5)

Last time: \( \vec{v} = (v_1, v_2, v_3) \), \( \vec{w} = (w_1, w_2, w_3) \)

\[ \vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 \]

Key: \( \vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos \theta \)

Projection:

\( \text{proj}_\vec{v} \vec{w} = \text{component of } \vec{w} \text{ along } \vec{v} \)

\[ = \text{scalar mult of } \vec{v} \text{ closest to } \vec{w} \]

Note: \( |\text{proj}_\vec{v} \vec{w}| = |\vec{w}| \cos \theta \)

So:

\[ \text{proj}_\vec{v} \vec{w} = |\vec{w}| \cos \theta \left( \frac{\text{unit vector pointing in same dir as } \vec{v}}{1 |\vec{v}|^2} \right) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|^2} \]
Work = (force) \times (distance)

\[ W = |\hat{F}||\hat{d}| \]

\[ = |\text{proj}_d \hat{F}||\hat{d}| \]

\[ = \left| \frac{\hat{F} \cdot \hat{d}}{|\hat{d}|^2} \right| |\hat{d}| \]

\[ = \hat{F} \cdot \hat{d} \]

\[ W = \hat{F} \cdot \hat{d} \]

**Regression:**

Roughly \( y = cx \).

What is \( c \)?

\( \hat{x} = (x_1, x_2, x_3, \ldots, x_n) \)

\( \hat{y} = (y_1, y_2, y_3, \ldots, y_n) \)
If \( y_i = c x_i \) for each \( i \), then
\[
\vec{y} = c \vec{x}
\]

So "best fit" is
\[
c = \frac{\vec{x} \cdot \vec{y}}{1 \vec{x}^2}
\]

which minimizes \( |\vec{y} - c \vec{x}| \) [in general]

[Model has more parameters, and proj is onto a plane or similar. Cf. Math 415.]

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Planes in \( \mathbb{R}^3 \):

Take a point \( P_0 = (x_0, y_0, z_0) \) on the plane, with \( \vec{r}_0 \) its position vector.

Test if \( P = (x, y, z) \) is in the plane:

\( \vec{n} \) and \( \vec{r} - \vec{r}_0 \) are perpendicular (orthogonal)

i.e. \( \vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \)
If \( \vec{n} = (a, b, c) \), have \( P = (x,y,z) \) in the plane when

\[
0 = (a,b,c) \cdot (x-x_0, y-y_0, z-z_0)
\]

\[
= a(x-x_0) + b(y-y_0) + c(z-z_0)
\]

\[
= ax + by + cz + d
\]

with \( d = -(ax_0 + by_0 + cz_0) \)

Conversely,

\[
a x + b y + c z + d = 0
\]

defines a plane [unless \( a = b = c = 0 \)].

[Normal vectors will be a key concept later in the course. Can be used to solve geometric problems about planes]
Example: $P_1$ is a plane given by $x + y + z - 1 = 0$.

Plane is set by 3-points:

Normal vector

$\vec{n}_1 = (1, 1, 1)$

$P_2 = \{z = 1\}$

$\vec{n}_2 = (0, 0, 1)$

Q1: What is the intersection of $P_1$ and $P_2$?

Q2: What is the angle between $P_1$ and $P_2$?
A line $L$:

points on $L$ sat

\[ x + y + z = 1 \quad \text{and} \quad z = 1 \]

Two easy solutions:

\[ \vec{r}_0 = (0, 0, 1) \]
\[ \vec{r} = (1, -1, 1) \]
\[ \vec{v} = \vec{r} - \vec{r}_0 = (1, -1, 0) \]

Every pt on $L$ has the form

\[ \vec{r}(t) = \vec{r}_0 + t \vec{v} \]
and so $L$ is parameterized by

$$(0,0,1) + t(1, -1, 0) = (t, -t, 1)$$

This is just like worksheet 3(e) from yesterday.