Last time:

\[ \mathbb{R}^3 \]

\[ (1, 3, 2) \]

\[ \mathbb{R}^n = \text{tuples} \]

\[ (x_1, x_2, \ldots, x_n) \]

of real numbers.

Vectors in \( \mathbb{R}^2 \): arrows where both direction and length are important.

\[ \hat{v} \]

\[ \hat{w} \]

\[ \hat{v} \] and \( \hat{w} \) are regarded as equal if they are the same up to translation.

\[ \hat{v} = \hat{w} \]
Addition: \[ \overrightarrow{V} \rightarrow \overrightarrow{W} \]

Start at A, head along \( \overrightarrow{V} \) to B, then along \( \overrightarrow{W} \) to C.

Note

\[ \overrightarrow{V} + \overrightarrow{W} = \overrightarrow{W} + \overrightarrow{V} \]

and so

Resulting force is the sum of the two forces.

Like points in \( \mathbb{R}^2 \), vectors are param. by pairs of #s.

\[ \overrightarrow{V} = (b_1 - a_1, b_2 - a_2) \]

\[ A = (a_1, a_2) \]

\[ B = (b_1, b_2) \]
Points and vectors are different, though.

Given \( \vec{V} = (V_1, V_2) \) and \( \vec{W} = (W_1, W_2) \),

then

\[ \vec{V} + \vec{W} = (V_1 + W_1, V_2 + W_2) \]

Scalar multiplication:

\[ c \vec{V} = \text{same direction} \]

\[ \text{length scaled by } c. \]

\[ \vec{V} \quad 2\vec{V} \quad \frac{1}{2} \vec{V} \]

Clue coordinates

\[ c \vec{V} = (cV_1, cV_2) \quad \text{where } \vec{V} = (V_1, V_2) \]
Length:
\[ |\vec{v}| = \sqrt{v_1^2 + v_2^2} \]

[Some books use \( \| \vec{v} \| \).]

Properties: [Can work out from geom or alg def.]
\[
\begin{align*}
\vec{v} + \vec{w} &= \vec{w} + \vec{v} \\
(\vec{u} + \vec{v}) + \vec{w} &= \vec{u} + (\vec{v} + \vec{w}) \\
\vec{v} + \vec{0} &= \vec{v} \quad \text{if} \quad \vec{0} = (0,0) \\
\vec{v} + (-1)\vec{v} &= \vec{0} \\
c(\vec{v} + \vec{w}) &= c\vec{v} + c\vec{w} \\
(c + d)\vec{v} &= c\vec{v} + d\vec{v} \\
(c \cdot \vec{v}) &= c(d\vec{v}) \\
\perp \vec{v} &= \vec{v}.
\end{align*}
\]

Vectors in \( \mathbb{R}^3 \): Same idea
\[
\vec{v} = (v_1, v_2, v_3)
\]
Can add etc. as before

Vectors in \( \mathbb{R}^n \): Once more, with feeling.
Standard vectors:

\[ \mathbb{R}^2 = \begin{array}{c}
\vec{i} = (1, 0) \\
(0,1)
\end{array} \]

\[ 3 \vec{i} + 2 \vec{j} = (3,0) + (0,2) = (3,2) \]

\[ \mathbb{R}^3: \]

\[ \vec{k} = (0,0,1) \]

\[ \vec{j} = (0,1,0) \]

\[ \vec{i} = (1,0,0) \]

\[ (1,3,2) = \vec{i} + 3 \vec{j} + 2 \vec{k} \]

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**Can we multiply vectors?** In \( \mathbb{R}^3 \), there are two ways to do this, different from each other and also ordinary mult of #s.

**Dot Product:** \( \vec{V} = (v_1, v_2, v_3), \vec{W} = (w_1, w_2, w_3) \) vectors in \( \mathbb{R}^3 \)

\[ \vec{V} \cdot \vec{W} = v_1 w_1 + v_2 w_2 + v_3 w_3 \]
\[ \text{Ex: } \vec{v} = (1, 2, 0), \quad \vec{w} = (-1, 0, 2) \]
\[ \vec{v} \cdot \vec{w} = 1 \cdot (-1) + 2 \cdot 0 + 0 \cdot 2 = -1 \]

**Key:**
\[ \vec{v} \]
\[ \vec{w} \]
\[ \theta \]
\[ \vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos \theta \]

\[ 0 \leq \theta \leq \pi \text{ is the smaller of two angles.} \]

\[ \text{Ex: } \cos \theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| |\vec{w}|} = -\frac{1}{5} \quad \Rightarrow \quad \theta \approx 101.54^\circ \]

**Note:** Thus
\[ \vec{v} \cdot \vec{w} = 0 \]

exactly when \( \cos \theta = 0 \), i.e. \( \theta = \pm \frac{\pi}{2} \) and the vectors meet at right angles.
Properties: [Easy to see from the def.]
\[
\begin{align*}
|\vec{v} \cdot \vec{v}| &= \vec{v} \cdot \vec{v} \\
\vec{u} \cdot (\vec{v} + \vec{w}) &= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \\
\vec{v} \cdot \vec{w} &= \vec{w} \cdot \vec{v} \\
(c \vec{v}) \cdot \vec{w} &= c (\vec{v} \cdot \vec{w})
\end{align*}
\]

Idea behind the Key formula:

Law of Cosines:
\[
C^2 = a^2 + b^2 - 2ab \cos \Theta
\]

\[
C^2 = |\vec{w} - \vec{v}| = (\vec{w} - \vec{v}) \cdot (\vec{w} - \vec{v})
\]

\[
= \vec{w} \cdot \vec{w} + \vec{v} \cdot \vec{v} - 2 \vec{v} \cdot \vec{w}
\]

\[
= a^2 + b^2 - 2 \vec{v} \cdot \vec{w} \implies \vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos \Theta.
\]