
Hasse-Minkowski: A quad form over $\mathbb{Q}$ reps 0 iff it does so at each place $\mathbb{Q}_v$.

Pf: $n \geq 5$, conduct on $n$. Can take
\[
g = a_1x_1^2 + a_2x_2^2 - (a_3x_3^2 + \cdots + a_nx_n^2), \quad a_i \in \mathbb{Z}^x
\]

Consider: reduce to $b \mathbb{Z}^2 - c$ where $f$ reps $a \in \mathbb{Q}^x$ and $g' = b \mathbb{Z}^2 - g$ reps 0 at every $v$.

[Point $g'$ reps 0 $\Rightarrow$ $g$ reps 0.] Let $S = \{2, \infty\} \cup \{\text{primes div } a_i\}$.

As $g_v$ reps 0, $\exists b_v \in \mathbb{Q}_v^x$ with $f(x_1^v, x_2^v) = b_v = g(x_3^v, \ldots, x_n^v)$

Let $f_v^{-1}(b_v(\mathbb{Q}_v^2)) = U_v \subseteq \mathbb{Q}_v^x \times \mathbb{Q}_v$, which is open.

By compactness, $\exists (x_1, x_2) \in \mathbb{Q}^2$ lying in $\prod_{v \in S} U_v \subseteq (\prod_{v \in S} \mathbb{Q}_v)^2$. Then $b = f(x_1, x_2)$ has the prop that $b = b_v c_v^2$ in $\mathbb{Q}_v$ for $v \in S$.

Thus, for $v \in S$, $g_v$ reps $b$ and so $g'_v$ reps 0.

For $v \notin S$, we saw last time that $g_v$ reps $b$ and hence $g'_v$ reps 0. By induction, $g'_v$ reps 0 over $\mathbb{Q}$.
\[ \Rightarrow \text{q reps 0 over } \mathbb{Q}. \]

Q.E.D.

Remarks: Simple to state yet the proof uses many things, such as quad reciprocity, Dirichlet's The...  

2. Encodes many classical results, e.g., every number is a sum of 4 squares. [Basically everything red to what happens mod 8...]

3. Key to understanding quaternion algebras.

\[ \text{Adèles and âdèles: } \mathbb{V} = \text{places of } \mathbb{Q} \]

Consider \[ \mathbb{Q} \rightarrow \prod_{\mathbf{v} \in \mathbb{V}} \mathbb{Q}_\mathbf{v} = \mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3 \times \mathbb{Q}_5 \times \ldots \]

\[ \times \longrightarrow (x, x, x, \ldots) \]

Image lies in this subsring, called the adèles

\[ \mathbb{A} = \{(x_\mathbf{v}) \in \prod_{\mathbf{v} \in \mathbb{V}} \mathbb{Q}_\mathbf{v} | x_\mathbf{v} \in \mathbb{Z}_\mathbf{v} \text{ for almost all } \mathbf{v} \} \]

Topologize as follows (not the prod. top!)

A basis about 0 consists of sets of the form \[ U = \prod \mathbb{U}_\mathbf{v} \text{ where } \mathbb{U}_\mathbf{v} \subseteq \mathbb{Q}_\mathbf{v} \text{ is an open set containing 0 and almost all } \mathbb{U}_\mathbf{v} = \mathbb{Z}_\mathbf{v} \]
More open sets than the prod. top., e.g. \[ U = (-1,1) \times \mathbb{Z}_v \] is open. However, the top restricted to \( U \) or \( \mathbb{A} = [-1,1] \times \prod_{v \neq \infty} \mathbb{Z}_v \) is just the prod. top. Note \( \mathbb{A} \) is compact, and in fact \( \mathbb{A} \) is a locally compact topological ring.

Prop: \( \mathbb{Q} \) is discrete in \( \mathbb{A} \), and \( \mathbb{A}/\mathbb{Q} \) is compact.

Compare: \( \mathbb{Z} \subset \mathbb{R} \) is discrete, \( \mathbb{R}/\mathbb{Z} = S^1 \)

\[ \mathbb{Q} \to \mathbb{K} \] \( \mathbb{R} \) is discrete with \( \mathbb{K}/\mathbb{Q} = (S^1)^n \)

In general, any number field has a ring of adeles.

Proof of discreteness: Enough to show \( \exists U \) open \( \subset \mathbb{A} \) with \( U \cap \mathbb{Q} = \emptyset \). Can take \( U = (-1/2,1/2) \times \prod_{v \neq \infty} \mathbb{Z}_v \), since if \( x \in \mathbb{Q}^* \), then \( x \in \mathbb{Z}_v \) for all \( v \), \( \Rightarrow x \in \mathbb{Z} \).

Compactness: \( U \to \mathbb{A}/\mathbb{Q} \).

Idea: Given \( a \in \mathbb{A} \), choose an elt of \( \mathbb{Q} \) so that its in \( \mathbb{R} \times \prod_{v \neq \infty} \mathbb{Z}_v \), then map by \( x \mapsto x \mathbb{Q} \in \mathbb{Z} \).
Topologically, \( A/Q = \lim_{\to} \mathbb{R}/n\mathbb{Z} \)
\[ \text{a solenoid (Cantor set bundle over } S^{1}) \]

Note: \( A/Q \) is the Pontryagin dual of \((\mathbb{Q},+)\)
needed to do Fourier analysis on \( \mathbb{Q} \).
(c.f. \( S^{1} \leftrightarrow \mathbb{Z} \) and \( \mathbb{R} \leftrightarrow \mathbb{R} \)).

A circle: \( \mathbb{T}_{K} = \mathbb{A}_{K}^{X} \) (with a slightly different topology)
\( \mathbb{K}^{X} \) is discrete in \( \mathbb{T}_{K} \), so consider the
idèle class group \( C_{K} = \mathbb{T}_{K}/\mathbb{K}^{X} \). This
isn't compact, but we set \( \mathbb{T}_{\infty} = \prod_{K_{V} \in \mathcal{V}_{\infty}} \mathbb{K}_{V}^{X} \times \prod_{K_{v} \in \mathcal{V}_{f}} \mathbb{O}_{v}^{X} \)

then
\[ C_{K} = \frac{\mathbb{T}_{K}}{\mathbb{K}^{X} \cdot \mathbb{T}_{\infty}} \]

\[ \text{usual ideal class group} \]