L/K normal, $\beta$ a prime of $O_K$.
$\text{Gal}(L/K)$ acts on \{primes of $O_L$\} $= \mathbb{Q}$ e.g.
have a hom. $\text{Gal}(L/K) \to \text{Sym}(\mathbb{Q})$

**Last time:** Action is transitive.

Will use this action to $L/K$ up into simpler pieces
from $\beta$'s point of view: $\mathfrak{p}_L = \mathfrak{p}_E^e \cdot \mathfrak{p}_D^f$ $\Rightarrow$ ram index inital degree

\[
\begin{array}{cccc}
\vdots & \mathfrak{p}_L & \mathfrak{p}_E & \mathfrak{p}_D \\
\text{intrinsic field} & L & E & D \\
\text{decomposition field} & \vdots & \mathfrak{p}_E & \mathfrak{p}_D \\
K & \beta & \vdots & \vdots \\
\end{array}
\]

Here $\mathfrak{p}_E = \mathfrak{p}_E \cap \mathfrak{O}_E$ and $\mathfrak{p}_D = \mathfrak{p}_D \cap \mathfrak{O}_D$

Set $G = \text{Gal}(L/K)$.

$D = \text{Decomposition group} = \{g \in G \mid g(\mathfrak{p}_E) = \mathfrak{p}_E, g(\mathfrak{p}_D) = \mathfrak{p}_D\}$

Stabilizer of $\mathfrak{p}_D$ under $G$-action
\[ E = \text{Inertia group } = \{ \sigma \in D \mid \sigma \text{ acts trivially on } \mathbb{Q}/\mathbb{Z} \} \]

Then \( L = \text{fixed field } \frac{E}{D} = \{ x \in L \mid \sigma(x) = x \text{ for all } \sigma \in E \} \)

\( L_D = \text{fixed field of } D \).

**Notation:** \((\mathfrak{p})_E = \text{thing fixed by } E\)

\((\mathfrak{p}_L)_E = \mathfrak{p}_L \cap L_E = \mathfrak{p}_{LE} \text{ and } \mathfrak{p}_E = \mathfrak{p}_D \cap \mathfrak{p}_{LE}\)

**Thm:** (i) \((\mathfrak{p}_D)_{L_D} \text{ is non-split in } L_D\)

\([\text{i.e., } \mathfrak{p}_D \text{ is the only prime above } \mathfrak{p}_D]\]

\( L \uparrow_{L_D} \mathfrak{p}_D \)

(ii) \( \mathfrak{p}_D \text{ over } \mathfrak{p}_D \text{ has ram index } = e \text{ internal degree } = f \)

(iii) \( \mathfrak{p}_D \text{ over } \mathfrak{p}_D \text{ has ram index } = 1 \text{ internal deg } = 1 \)

\( G_D = \text{cl} \mathfrak{p}_D, \beta \text{ breaks up into } r \text{ pieces (same as in } L) \)

\( \text{unramified, with residue field } (\cong \mathbb{Q}_L/\mathfrak{p}_D \cong \mathbb{Q}_K/\beta) \)

\( \text{unchanged. Also } [L_D : K] = r. \text{ That is, } \beta \text{ is totally split in } L_D.\)

**Proof:** (i) \( \text{Gal}(L/L_D) = D \text{ acts transitively on the primes above } \mathfrak{p}_D \text{ yet fixes } \mathfrak{p}_D. \)

L/K is normal, hence so is \( L/L_D. \)
As \( L/K \) is normal, we have

\[ n = [L:K] = \sum e_i f_i = e f r. \]

Since \(|D| = |G|\), we have \(|D| = e f r\). Thus, by (ii) \( \beta \) must split into \( L/D \) factors in \( G_L \).

The fund. evident forces

\[ e'' = f'' = 1. \]

By mult. of ram index + inertial deg,

get \( e' = e \) and \( f' = f \).

Consider the residue fields

\[
K(\sigma) = \frac{G_L}{\sigma g}
\]

\[
K(\beta) = \frac{G_K}{\beta}
\]

Prop: This extension is normal, and \( D \rightarrow \text{Gal}(K(\sigma)/K(\beta)) \) is surjective.

Proof: By the above, \( K(\beta) \cong K(\sigma_0) \) so we might as well assume \( L_D = K \Rightarrow D = G \).

Blah, blah, ...