Lecture 30: Quadratic Forms I

Throughout, $K$ is a field of char $\neq 2$.

**Quadratic Space**: a finite dim'l vector space $V$ over $K$ with a symmetric bilinear form $B: V \times V \to K$.

\[ B(v, w) = \sum v_i w_i \]

\[ B(kv_1 + v_2, w) = k B(v_1, w) + B(v_2, w) \]

**Ex:** $V = \mathbb{R}^n$ and $B(v, w) = \sum v_i w_i$

**Ex:** $V = \mathbb{Q}^2$ \quad $B(v, w) = 8v_1 w_1 + v_1 w_2 + v_2 w_1 - 3v_2 w_2$

**Quadratic Form**: $g: V \to K$ given by $g(v) = B(v, v)$

**Note**: $g$ actually determines $B$ via

\[ B(v, w) = \frac{1}{2} (g(v + w) - g(v) - g(w)) \]

**Ex:**

1. $g = v_1^2 + \ldots + v_n^2$
2. $g = 8v_1^2 + 2v_1 v_2 - 3v_2^2$

**Note**: $g(v)$ can be negative, or even $0$, e.g. in case 2

$g((0, 1)) = -3$ and $g((1, 2)) = 8 + 4 - 3.2^2 = 0$

$B$ is determined by its Gramm matrix $G = (B(e_i, e_j))$

with respect to any basis $B = \{e_1, \ldots, e_n\}$ of $V$.

In particular, if $x, y \in V$, then
column vector cor to \( w \) in basis \( B \)

\[
B(v, w) = [v]_B^T C [w]_B = (a_1, a_2, \ldots, a_n) \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}
\]

\[
= \sum_{i,j} a_i B(e_i, e_j) b_j = B(v, w) \quad \text{as} \quad v = \sum a_i e_i \\
\text{w} = \sum b_i e_i
\]

If \( B' \) is another basis of \( B \), then

\[ G_{B'} = C^T G_B C \quad \text{where} \quad C = \begin{bmatrix} \text{Input} \B' \\ \text{Output} B \end{bmatrix} \]

\underline{Discriminant:} \( \det G_B \), well-defined up to mult by \((k^*)^2\).

\[ \begin{align*}
\text{Ex:} & \quad \text{\( G = \text{Id} = (1,0) \) so \( \det = 1 \)} \\
\text{\( G = (8,1) \) so \( \det = -25 \equiv -1 \mod (k^*)^2 \)}
\end{align*} \]

While the only cond. on \( G \) that \( x^T G y \) define a bilinear form is that \( G^T = G \), we'll show:

\[ (V, B) \text{ a quad space. Then } \exists \text{ a basis } B \text{ for } V \text{ where } G_B \text{ is diagonal, i.e. } g(v) = \sum q_i v_i^2. \]

\[ \text{Ex2: Take } e_1 = (0,1). \text{ If } G \text{ is to be diagonal, then need } B(e_1, e_2) = 0 \iff w_1 - 3w_2 = 0 \text{ where } e_2 = (w_1, w_2) \text{ so take } e_2 = (3,1). \text{ Then } G = \begin{pmatrix} -3 & 0 \\ 0 & 7 \end{pmatrix} \]

(c.f. Gramm-Schmidt)
Def: For a subset $W$ of $V$, let

$$W^+ = \{ v \in V \mid B(v, w) = 0 \text{ for all } w \in W \}$$

If $v \in V$ has $B(v, v) \neq 0$, then $V = \langle v \rangle \oplus \{ v \}^+$. Since

$$\dim \{ v \}^+ = \dim V - 1 \text{ as } \{ v \}^+ = \ker (v^T C)$$

1×n matrix

Pf of Thm: By induction, we find an orthogonal direct sum

$$V = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \langle e_3 \rangle \cdots \oplus \langle e_k \rangle \oplus W$$

where $g(w) = 0$ for all $w \in W$. Since $g$ determines $B$, it follows that $B(w_1, w_2) = 0$ for all $w_1, w_2 \in W$. So completing $\{ e_1, \ldots, e_k \}$ to a basis of $V$ using elements of $W$ gives the needed basis.

Def: $B$ is nonsingular if every $v \neq 0$ in $V$ has some $w \in W$ with $B(v, w) \neq 0$.

Equivalent formulations:

1. $V^L = \mathbb{R}^k$.

2. $V \rightarrow V^*$ is an isomorphism $V \rightarrow B(v, \cdot)$

3. $\det C \neq 0$.

Always have $V = V^+ \oplus W$ where $B|_W$ is nonsingular.
Prop: If \( B \) is non-singular, then for every subspace \( W \)

1. \((W^+)\) = \( W \)
2. \( \dim W + \dim W^\perp = \dim V \)

Note: Need not have \( V = W \oplus W^+ \) as sometimes \( W \cap W^+ \neq \{0\} \), e.g. \( w \neq 0 \) in \( W \) with \( g(w) = 0 \).

Pf: HW.

Classification of non-singular quad forms, up to isometry:

- **\( K = \mathbb{C} \):** only one, namely example (1).
- **\( K = \mathbb{R} \):** there are \( \dim V \) of them, namely

  \[ B_k = x_1^2 + \cdots + x_k^2 - (x_{k+1}^2 + \cdots + x_n^2). \]

  Signature = \# pos - \# neg

  = \( 2k - n \)

Pf: Can choose a basis \( \{e_1, e_2, \ldots, e_n\} \) where

\( G = \text{diag}(d_1, \ldots, d_n). \) Replacing \( e_i \) with \( \lambda e_i \) changes \( d_i \) to \( \lambda^2 d_i. \)

Things are more complicated with \( K = \mathbb{Q} \).

E.g. in example 2, \( G = \begin{pmatrix} 8 & 1 \\ 1 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 0 \\ 0 & 75 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \)

In fact, \( G \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \)

as if \( C = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \) then \( C^2 \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix} C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \)

On the other hand, \( x_1^2 - 3x_2^2 \) is really different than \( x_1^2 - x_2^2 \) because of the disc...