Math 530: Problem Set 7

Due date: In class on Wednesday, April 8.
Course Web Page: http://dunfield.info/530

1. In this problem, you’ll consider the connection between the discriminant and the degree of a number field.
   (a) Show that the only number field \( K \) with \(|\Delta_K| = 1\) is \( \mathbb{Q} \). An immediate consequence is that for any number field \( K \neq \mathbb{Q} \) there is some rational prime that ramifies in \( \mathcal{O}_K \).
   (b) Show that \(|\Delta_K| \to \infty \) as \([K : \mathbb{Q}] \to \infty\).

2. Let \( K \) be a number field and \( a \subset \mathcal{O}_K \) an ideal.
   (a) Prove there exists a finite extension \( L \) of \( K \) so that \( a\mathcal{O}_L \) is principle.
   (b) Does there always exists a finite extension \( L \) in which every ideal of \( \mathcal{O}_K \) becomes principle in \( \mathcal{O}_L \)? Prove your answer.

3. For a number field \( K \), the order of the ideal class group is called the class number and usually denoted \( h \). Show that the quadratic fields with discriminant 5, 8, 12, \(-3\), \(-4\), \(-7\), \(-8\), \(-11\) have class number 1.

4. Show that the class groups of \( \mathbb{Q}(\sqrt{10}) \) and \( \mathbb{Q}(\sqrt{-10}) \) are both \( \mathbb{Z}/2\mathbb{Z} \).

5. Compute the class group of \( \mathbb{Q}(\sqrt[3]{7}) \).

6. Let \( K \) be a totally real number field, i.e. the image of every embedding \( K \to \mathbb{C} \) is contained in \( \mathbb{R} \). Let \( T \) be a proper nonempty subset of the set of embeddings \( \tau: K \to \mathbb{R} \). Prove there exists a unit \( \epsilon \in \mathcal{O}_K^\times \) satisfying \( 0 < \tau(\epsilon) < 1 \) for \( \tau \in T \) and \( \tau(\epsilon) > 1 \) for \( \tau \notin T \).
   **Hint:** Apply Minowski’s lattice point theorem to the image of the unit lattice in the trace-zero subspace.