Math 530: Final Problem Set.1
All quaternions, all the time.

Due date: In class on Wednesday, May 6.
Reminder: Our final will be on Thursday, May 14 from 8-11am in our usual classroom.

1. Let $K$ be a field of characteristic $\neq 2$, and choose $a, b \in K^\times$. Consider the following associative algebra over $K$: let $A$ be the $K$ vector space with basis $\{1, i, j, k\}$ and multiplication determined by $i^2 = a, j^2 = b$, and $ij = -ji = k$. The algebra $A$ is called a quaternion algebra, and these are very important examples in number theory. The algebra $A$ is sometimes denoted by its Hilbert symbol $\left( \frac{a, b}{K} \right)$. For instance, Hamilton's original, accept no substitutes, quaternions are $\mathcal{H} = \left( \frac{-1,-1}{\mathbb{R}} \right)$.

(a) Prove that $M_2(K)$, the algebra of $2 \times 2$ matrices is a quaternion algebra. Hint: It’s $\left( \frac{1,1}{K} \right)$.
(b) Different Hilbert symbols can give rise to isomorphic quaternion algebras. Give an example.
(c) Prove that the only quaternion algebra over $\mathbb{C}$ is $M_2(\mathbb{C})$ and the only two over $\mathbb{R}$ are $M_2(\mathbb{R})$ and $\mathcal{H}$.

2. Let $A = \left( \frac{a,b}{K} \right)$. For $\alpha = w + xi + yj + zk$, define its conjugate to be $\overline{\alpha} = w - xi - yj - zk$. Then we can define the norm $N: A \to K$ by $N(\alpha) = \alpha \overline{\alpha}$ and trace $tr: A \to K$ by $\alpha + \overline{\alpha}$.

(a) Calculate the norm and trace explicitly for $\mathcal{H}$.
(b) Show that $N$ gives a quadratic form on $A$, which is diagonal with respect to the standard basis $\{1, i, j, k\}$.
(c) Show that the norm and trace are multiplicative and additive, respectively. Hint: There's a nice formula for $\alpha \beta$, which typically isn't equal to $\overline{\alpha \beta}$.
(d) What standard quantities are the norm and trace on $M_2(K)$?
(e) The subspace $A_0 = \{ \alpha \in A \mid tr(\alpha) = 0 \}$ is called the pure quaternions. For Hamilton's quaternions, we have $\mathcal{H}_0 \cong \mathbb{R}^3$. Prove that quaternion multiplication on $\mathcal{H}_0$ is a combination of the usual dot and cross products as follows: $\alpha \beta = \alpha \times \beta - \alpha \cdot \beta$. This is the source of the convention in vector calculus that the standard basis of $\mathbb{R}^3$ is $\{i, j, k\}$.

3. Recall that an algebra is a division algebra if every nonzero element has a multiplicative inverse.2 Let $A = \left( \frac{a,b}{K} \right)$. It is not hard to show that $A$ is a central simple algebra over $K$; thus by Wedderburn's theorem it is either $M_2(K)$ or a division algebra. Prove that the following are equivalent:

(a) $A \cong M_2(K)$; equivalently, $A$ is not a division algebra.
(b) The norm form on $A$ has an isotropic vector.
(c) The norm form on $A_0$ has an isotropic vector.
(d) The usual Hilbert symbol $(a, b) = 1$.

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1Revised May 1, 2009 to fix problem 2(e).
2A synonym for division algebra is “noncommutative field” which is a good way to think about such objects.