Lecture 7: Applications of \( \pi_1 \)

Last time: \( \text{Thm } \pi_1 S^1 = \mathbb{Z}, \quad \pi_1(\infty) = \text{FreeGroup}(a,b) \)

[Soon will give Van Kampen's Thm, let us compute \( \pi_1 \).]

Today:

Fund Thm of Algebra: Every non-constant poly \( p(z) \in \mathbb{C}[z] \) has a root in \( \mathbb{C} \).

Brouwer Fixed Point Thm: For every cont. map \( h : D^2 \to D^2 \), there is an \( x \in D^2 \) with \( f(x) = x \).

Induced maps: \( h : (X,x_0) \to (Y,y_0) \) gives \( h_* : \pi_1(X,x_0) \to \pi_1(Y,y_0) \) via \( [f] \longrightarrow [h \circ f] \).

Ex. \( h : S^1 \longrightarrow S^1 \)

\( Z \longrightarrow \mathbb{Z}^2 \)

\( b \quad y_0 \quad \text{Then} \)

\( h_* : \pi_1(S',y_0) \to \pi_1(S',x_0) \)

\( \langle a \rangle \quad b \longrightarrow a^2 \)

\( h_* \) is a group homomorphism.
Claim: There does not exist \( r: D^2 \rightarrow S' \) with \( r|_{S'} = \text{id}_{S'} \). Hence an \( r \) no exists to \( S' \).

\[ r|_{S'} = \text{id}_{S'} \]

\[ f(x) = t(x - h(x)) + x \text{ where } t \geq 0 \]

Continuity. [\( F \) is continuous, \( F(0) = 0 \).]

Picture at left. Write \( r \) is

\[ f: [0,1] \rightarrow D^2 \text{ has no root.} \]

\[ f(0) = 0 \text{ and } f(1) = 1 \]

\[ g: [0,1] \rightarrow D^2 \text{ has no root.} \]

\[ g(0) = 0 \text{ and } g(1) = 1 \]
**Proof 1:** Consider the map $S^1 \to S^1$ which is the identity. By assumption, this extends to a map $D^2 \to S^1$. By H.W., we know that must thus be trivial in $\pi_1 S^1$, a contradiction since it generates $\pi_1 S^1 = \mathbb{Z}$.

**Proof 2:** The map $\tilde{r}_* : \pi_1(D^2, 1) \to \pi_1(S^1, 1)$ has trivial image, since $\tilde{r} = 0$. If $f : [0, 1] \to S^1$ given by $s \mapsto e^{-2\pi si}$, then $\tilde{r}_*([f]) = [\text{id} \circ f] = [f]$ which generates $\pi_1(S^1, 1)$, a contradiction.

**Proof of the F.T.A.:** Let $p(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_0$ be any polynomial. Let $S^1_r = \{ z \in \mathbb{C} \mid |z| = r \}$, $Y = \mathbb{C} \setminus \{0\}$. For $r$ large we have $f_r : S^1_r \to Y$ given by $f_r = p|_{S^1_r}$. 

\[\begin{array}{ccc}
S^1 & \xrightarrow{f_r} & Y
\end{array}\]
Claim 1: \( \pi_1 Y = \mathbb{Z} \), where by winding number.

Claim 2: The map \( \pi_1(S^1_r) \to \pi_1(Y) \) is mult by \( n \).

[Argue claim 1 is plausible, as is claim 2 if you consider \( a_k = 0 \).]

Assume \( p \) has no roots. Then

\[
\begin{array}{ccc}
S_r^1 & \xleftarrow{i} & C \\
\downarrow{f_r} & & \downarrow{p} \\
Y & \xrightarrow{=} & Y
\end{array}
\]

and so

\[
\pi_1(S^1_r) \xrightarrow{i_*} \pi_1(C) \xrightarrow{p_*} \pi_1(Y)
\]

Note that \( p_* \circ i_* = f_{r_*} \), but the first is the \( 0 \) map and the other mult by \( n \). Thus \( n = 0 \) on \( p \) is constant.

Formalization: Consider \( f_r : I \to S^1_r \) given by

\[
f_r(s) = \frac{p(re^{-2\pi si})/p(r)}{|p(re^{-2\pi si})/p(r)|} \quad \text{which makes sense if } p \text{ has no zeros.}
\]

This is a loop at \( I \), and equal to \( 0 \) in \( \pi_1 S^1_r \) (take \( r \to 0 \)).

O.T.O.H., when \( r \) is large, see that \( [f] = n [\text{gen}] \).