**Lecture 6: Computing \( \pi_1 \) via the lifting correspondence**

**Last time:**

**Thm:**

\[ \tilde{f} : I \rightarrow \tilde{X} \]

\[ f \text{-path starting at } \tilde{x}_0 \]

For each \( \tilde{x}_0 \in p^{-1}(x_0) \), there exists a unique lift of \( f \) to a path starting at \( \tilde{x}_0 \).

Moreover, if \( g \circ p = f \) and \( \tilde{g} \) is its lift starting at \( \tilde{x}_0 \), then \( \tilde{g} \circ \tilde{f} \). In particular, \( \tilde{g}(1) = \tilde{g}(0) \).

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**Lifting correspondence:**

\[ p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0) \text{ covering map} \]

\[ \Phi : \pi_1(X, x_0) \rightarrow p^{-1}(x_0) \]

\[ [f] \mapsto \tilde{f}(1) \text{ where } \tilde{f} \text{ is the lift of } f \text{ starting at } \tilde{x}_0. \]

**Thm:** If \( \tilde{X} \) is simply connected, then \( \Phi \) is a bijection.

**Proof:**

\[ \Phi([a]) = \tilde{x}_1 \]

\[ \Phi([b]) = \tilde{x}_0 \]

\[ \Phi([a \cdot b \cdot a]) = \tilde{x}_0. \]
onto \( \tilde{\mathcal{X}} \in P^-(\mathcal{X}_0) \). As \( \tilde{\mathcal{X}} \) is path connected, \( \exists \) a path \( \tilde{f} \) from \( \tilde{\mathcal{X}_0} \) to \( \tilde{\mathcal{X}} \), then \([p \circ \tilde{f}] \in \pi_1(X, \mathcal{X}_0)\) and 
\[
\overline{\Phi}([p \circ \tilde{f}]) = \tilde{\mathcal{X}_1}.
\]

1-1: Suppose \( \overline{\Phi}([f]) = \overline{\Phi}([g]) \). Since \( \pi_1(\tilde{\mathcal{X}}, \tilde{\mathcal{X}_0}) = 1 \), we have 
\( \tilde{f} \simeq_p \tilde{g} \) via a path homotopy \( F \).

\[
([\tilde{f} \cdot \tilde{g}] = [\text{const } \tilde{\mathcal{X}_0}] \quad \text{and}
\]

\[
\text{hence } [\tilde{f}] = [\tilde{f}] \cdot [\tilde{g}] \cdot [\tilde{g}] = [\tilde{g}]
\]

Then \( p \circ F \) is a path homotopy from \( f \) to \( g \), i.e. \( [f] = [g] \).

Thm: \( \pi_1(S^1) = \mathbb{Z} \)

Ref: Consider the usual covering map \( p: \mathbb{R} \to S^1 \).
Claim: The bijection $\Phi: \pi_1(S^1, 1) \rightarrow \mathbb{Z}$ is a group homomorphism (hence isomorphism).

Note $\Phi([a^n]) = n$, and so $\Phi$ is a homomorphism restricted to the cyclic gp $\langle a \rangle$. As $\Phi(\langle a \rangle)$ is onto, we conclude $\pi_1(S^1) = \langle a \rangle \cong \mathbb{Z}$. $lacksquare$

Thm: For $X = \infty$ we have $\pi_1 X = \text{Free Group}(a, b)$.

Here:

$$\text{Free Group}(a, b) = \{ \text{words in symbols } a, a^{-1}, b, b^{-1} \setminus \text{not containing } aa^{-1}, a^{-1}a, bb^{-1}, b^{-1}b \}$$

$$= \{ a, b, aba^{-1}b^{-1}a, aba^{-1}b^{-1}a \ldots \}$$

with group operation "concatenate + cancel":

$$(aba^{-1}b^{-1}a) \cdot (a^{-1}bba) = aba^{-1}b^{-1}a^{-1}bb a \leftrightarrow \text{invalid string}$$

$$= aba^{-1}b a \leftrightarrow \text{final answer.}$$

Check associativity, inverses given by reverse and replace

$$(aba^{-1}bb)^{-1} = b^{-1}b^{-1}a^{-1}b a^{-1} a \leftrightarrow a^{-1} b \leftrightarrow b^{-1}$$

"Largest group generated by two elts."
Proof: Consider the covering space of $X$ which is the infinite 4-valent tree with vertices labelled according to the following rule:

Note $X$ is simply connected, and the lifting correspondence with $\tilde{X}_0 = e$ gives a bijection $\Phi : \pi_1(X, x_0) \rightarrow \text{FreeGroup}(a, b)$.

Check: This is a group homomorphism (illustrate with the product of $a^{-1}b$ and $ab$)

- Relate tree structure to associ of mult in free group
- Formal def of $\tilde{X} = \{\text{Vertices} = \text{elts of FreeGroup}(a, b), \text{Edges} = a\text{-edge from } w \text{ to } w a, b\text{-edge from } w \text{ to } w b\}$