Lecture: Covering transformations and regular covers.

Recall:

Covering transformations:

\[ \tilde{X} \xrightarrow{f} \tilde{X} \]

\[ \tilde{p} \circ f = p \]

\[ G(\tilde{X}) = \text{group of covering trans} \]

[\( \tilde{p} \) is comp of fun.] [really assoc to \( p \).]

\[ G(\tilde{X}) = \text{trans w\ integer shifts} \]

\[ \cong \mathbb{Z} \]

\[ G(\tilde{X}) = \{ \text{id} \} \]

[Query]

\[ a \]

[Ex. 3]

\[ a \rightarrow reb \rightarrow b \]

\[ a \rightarrow \infty \rightarrow b \]

\[ G(\tilde{X}) = \mathbb{Z}/2\mathbb{Z} \]
**Def:** A connected cover is normal, or regular, if \( \forall \tilde{x}_0, \tilde{x}_1 \in \tilde{X} \) with \( p(\tilde{x}_0) = p(\tilde{x}_1) \), \( \exists f \in G(\tilde{X}) \) with \( f(\tilde{x}_0) = \tilde{x}_1 \). [Ex: \( 1 \circ \sigma \) but not \( \sigma \)]

**Note:** For reasonable \( X \), such \( f \) exists \( \iff \)

\[
P_* \left( \pi_1 (\tilde{X}, \tilde{x}_0) \right) = P_* \left( \pi_1 (\tilde{X}, \tilde{x}_1) \right) = \gamma \cdot P_* \left( \pi_1 (\tilde{X}, \tilde{x}_0) \right) \cdot \gamma^{-1}
\]

**Thm:** \( X \) path conn, loc. path conn, S.L.S.C. Then a connected cover \( \tilde{X} \rightarrow X \) is regular \( \iff P_* \left( \pi_1 (\tilde{X}, \tilde{x}_0) \right) \) is a normal subgp of \( \pi_1 (X, x_0) \).

**Q:** What is \( \pi_1 (X, x_0) / P_* (\pi_1 (\tilde{X}, \tilde{x}_0)) \)?

\[
(\tilde{X}, \tilde{x}_0) \mathrel{P} (X, x_0) \) a regular cover. Given \( \alpha \in \pi_1 (X, x_0) \), let \( \tau_2 \) be the unique elt st. \( \tau_2 (\tilde{x}_0) = \tilde{x} \) (1) where \( \tilde{x}(0) = \tilde{x}_0 \).
Note: \( \tau_x \circ \beta = \tau_x \circ \tau \beta \)

and

\[ \tau_x = id \iff \alpha \in P_*(\pi_1(\tilde{X}, \tilde{x}_0)) \]

Thm: \( \tilde{X} \) a path conn. regular cover. Then

\[ \tilde{\tau} : \pi_1(X, x_0) / P_*(\pi_1(X, \tilde{x}_0)) \to \Gamma(\tilde{X}) \]

is an isomorphism.

Proof: Auto: \( \tilde{X} \) path conn. 1-1; obs. above.

Ex: \( \tilde{X} \) a universal cover of \( X \), get

\[ \pi_1(X, x_0) = \Gamma(\tilde{X}) \] as \( \langle 1 \rangle \) is normal.

Moreover, \( X = \tilde{X} / \Gamma(\tilde{X}) = \pi_1 \tilde{X} \).
$\mathbb{E}_X$: $S^1 = \mathbb{R}/\mathbb{Z}$, $\mathbb{C} = \mathbb{R}^2/\mathbb{Z}^2$, $\mathbb{RP}^2 \cup \mathbb{RP}^2 = \ldots$

$D_\infty = \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}$

etc...

Also: if $H \leq \pi_1(X, x_0)$ then the cover cor. to $H$ is $\tilde{X}/\sim(H)$.

$\mathbb{E}_X$: $S^1 \xrightarrow{p} S^n$ from $\mathbb{Z} \xrightarrow{1} \mathbb{Z}^3 \xrightarrow{\text{glue}} R/3\mathbb{Z} \rightarrow R/\mathbb{Z}$

Important note:

$\tau: \pi_1(X, x_0) \rightarrow G(\tilde{X})$ gives an action of $\pi_1$ on $p^{-1}(x_0)$. This has nothing to do the lifting correspondence from Monday.