Lecture 17: Classifying Covering Spaces

Last time:

**Theorem:** Covering spaces of a path conn, loe. path
conn space \( X \) are isomorphic via a \( f \) taking \( \tilde{x}_1 \) to \( \tilde{x}_2 \)
iff
\[ P_1 \left( \pi_1 \left( \tilde{x}_1, \tilde{x}_1 \right) \right) = P_2 \left( \pi_1 \left( \tilde{x}_2, \tilde{x}_2 \right) \right) \]

Mention survey results

**Theorem:** \( X \) path conn, loe. path conn, S.L.S.C.

\[ \left\{ \text{isom. classes of} \right\} \xleftrightarrow{\text{bijection}} \text{Subgroups of} \pi_1 \left( X, x_0 \right) \]

Note: The L.H.S. is classes of covers w/ choice of \( \tilde{x}_0 \) in \( p^{-1}(x_0) \)
\[ \begin{split}
\pi_1(\tilde{X}, \tilde{x_0}) & \ni b \text{ but not } a \\
\pi_1(\tilde{X}, \tilde{x_1}) & \ni a \text{ but not } b \\
\text{Notice that } \pi_1(\tilde{X}, \tilde{x_0}) & \text{ does contain } \pi_1(Y, a \cdot \tilde{y}) \\
& = abab^{-1}a^{-1}, \text{ and in fact} \\
\pi_1(Y) & = \pi_1(\tilde{X}, \tilde{x_0}) \\
& = \pi_1(\tilde{X}, \tilde{x_0})^{-1} \\
\end{split} \]

Then: \( X \) as above, then

\[
\begin{array}{ccc}
\text{iso class of} & \text{bijection} & \text{conj classes of subgps} \\
\text{path cong. classes } \tilde{x} \rightarrow X & & \text{of } \pi_i X \\
\end{array}
\]

Application: Subgps of free gps are free

(Quere class about this.)
Action on fibers:

\[ p: \tilde{X} \to X \text{ acts on } \alpha \in \pi_1(X, x_0). \]

Define \( L_\alpha \in \text{Sym}(p^{-1}(x_0)) \) by

\[ L_\alpha(\tilde{x}) = \tilde{x}(0) \]

where \( \tilde{x} \) is the lift of \( \alpha \) ending at \( \tilde{x} \).

\[ \text{Key: } L_\alpha \cdot \beta = L_\alpha \circ L_\beta \]

\[ \Rightarrow L_\alpha \text{ is really a bijection and } L : \pi_1(X, x_0) \to \text{Sym}(p^{-1}(x_0)) \text{ is a homomorphism.} \]

[Query:] \( \text{Stab}(\tilde{x}) = p_*(\pi_1(\tilde{X}, \tilde{x})) \quad L_\alpha(\tilde{x}) \)

[Query:] \( \tilde{X} \) is path \( \iff \) the action is transitive

\[ \pi_1(\infty) \to S_3 \]

\[ a \mapsto (23) \quad b \mapsto (12) \]

\[ L_\alpha(1) = 1 \]

\[ L_\alpha(2) = 3 \]

\[ L_\alpha(3) = 2 \]
Then: X path conn, i.e. path conn, S.L.S.C.

\[
\begin{align*}
\left\{ \text{Remn. covers of } X \text{ with } n \text{-sheets} \right\} & \leftrightarrow \left\{ L : \pi_i X \to S_n \text{ w/ trans invage mod conj in } S_n \right\} \\
\end{align*}
\]

Recall: Covering trans \( \tilde{X} \xrightarrow{f} X \)
\( p \downarrow \quad \downarrow p \)
\( X \)

\( f \) a homeo with \( p \circ f = p \).

\( G(\tilde{X}) = \text{ group of such } \)
\( \text{(op. is comp. of fns)} \)

\[ E \]
\[ \begin{array}{c}
\downarrow p \\
\circ \\
\end{array} \\
\]

\[ E \] \quad \begin{array}{c}
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\downarrow p \\
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\]

\[ E \] \quad [Query:] \( G(\tilde{X}) = \{ \text{id}_{\tilde{X}} \} \)

\[ E \] \quad \begin{array}{c}
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\downarrow p \\
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\end{array}
\]

\[ G(\tilde{X}) = \mathbb{Z}/2\mathbb{Z} \]
Def: A connected cover is normal (or regular, or Galois) if \( \forall \tilde{x}_0, \tilde{x}_1 \in \tilde{X} \) with \( p(\tilde{x}_0) = p(\tilde{x}_1) \),

\[ \exists f \in G(\tilde{X}) \text{ with } f(\tilde{x}_0) = \tilde{x}_1. \]

[Ex: \( \mathbb{R} \to \mathbb{S}^1, 0 \to \infty \)]

Note: For \( X \) reasonable, such \( f \) exists if

\[ n' \quad p_* (\pi_1 (\tilde{X}, \tilde{x}_0)) = p_* (\pi_1 (\tilde{X}, \tilde{x}_1)) \]

\[ = \gamma \quad p_* (\pi_1 (\tilde{X}, \tilde{x}_1)) \gamma'^{-1} \]

Thus: \( X \) path conn., loc. path conn., S.L.S.C.

A connected cover \( \tilde{X} \to X \) is regular

\[ \iff p_* (\pi_1 (\tilde{X}, \tilde{x}_0)) \text{ is a normal subgroup} \]

of \( \pi_1 (X, x_0) \).

Next time: \( \pi_1 (X) / p_* (\pi_1 \tilde{X}) \cong G(\tilde{X}) \)