Reminder: Exam Friday

Note: Can bring 1 sheet of paper to the exam.

$\pi_i$: fine as far as it goes, but need "higher dim'l" invariants. [Can't tell $S^2$ from $S^3$, let by $X^{(2)}$ etc.]

Higher homotopy groups:

$\pi_n (X, x_0) = \text{homotopy classes of maps}$

$$(S^n, s_0) \rightarrow (X, x_0)$$

[ homotopies pres. basept.]

Operation: for $\pi_1$:

for $\pi_2$:

\[ f \circ g \]

[Same for \[ \pi_n \].]
Like $\pi_1$, the group $\pi_n$ depends only on $X$ up to homotopy.

Problem: Hard to compute

[the flip side is that it contains a lot of info about the space]

Another problem with $\pi_1$ is that it's hard to tell if two finitely-presented groups are the same.

Ex: \[ \begin{array}{c|c} \text{ab} & \neq \\ \hline \text{abc} & \\end{array} \]

$\pi_1: \langle a, b | aba^{-1}b^{-1} = 1 \rangle \quad \langle a, b, c, d | aba^{-1}b^{-1} c d c^{-1} d^{-1} = 1 \rangle$

$\pi_1^{ab} / \langle [\pi_1, \pi_1] \rangle = \mathbb{Z}^2 \text{ vs. } \mathbb{Z}^4$. 

\[ \pi_1 \times \pi_1 \times \cdots \times \pi_1 \text{ (gen by id)} \]

\[ \pi_2 S^2 = \mathbb{Z} \]

\[ \pi_3 S^2 = \mathbb{Z} \]

\[ \pi_4 S^2 = \mathbb{Z} \]

\[ \pi_5 S^2 = \mathbb{Z} \]

\[ \pi_6 S^2 = \mathbb{Z} \]

\[ \vdots \]

\[ \exists \text{ an algorithm to compute } \pi_n S^m \]
Homology:

$H_n(X) = n$ dimensional things w/o boundary

$E_X: H_n(X) = \mathbb{Z}$ (# of path comps)

$\text{typical elt of } H_0$

Points have "signs": $+_0 + _0 = +_2 \Rightarrow H_0(X) = \mathbb{Z}_2$

$E_X: H_1(X) = \pi_1(X)^{ab}$

$\text{aba}^{-1} \in \pi_1 X = 0$

After abelianization

$\gamma \in \pi_1 X$ get $c_1, c_2, c_3$ come to how many times we cross each edge with sign
Not all triples appear
Thus \( c_1 + c_2 + c_3 = 0 \).
This is sufficient, too

Thus \( \pi_1^{ab} = \frac{\text{triples} (c_1, c_2, c_3)}{c_1 c_2 + c_3 = 0} = \frac{\mathbb{Z}^3}{c_1 + c_2 + c_3 = 0} = \mathbb{Z}^2 \)

\( X \) a space with a cell decomp.
\[ n \text{-chains: } C_n(X) = \text{free abelian gp gen by the } n\text{-cells.} \]
\[ \ell_X: X = \bigcirc \quad C_0(X) = \mathbb{Z} \oplus \mathbb{Z} = \{ a_0 x_0 + a_1 x_1 \} \]
\[ C_1(X) = \mathbb{Z}^3 = \{ c_1 e_1 + c_2 e_2 + c_3 e_3 \} \]
\[ C_n(X) = 0 \text{ for } n > 1. \]

\underline{Boundary map: } \partial_n : C_n(X) \rightarrow C_{n-1}(X) \text{ a homomorphism.}
\[ \partial_1 : C_1(X) \rightarrow C_0(X) \quad \partial_1(e_1) = x_1 - x_0 \]
\[ \partial_1(e_i) = x_1 - x_0 \]
Cycles (things w/o boundaries)

\( c \in C_n(X) \) with \( \partial c = 0 \), i.e. \( \ker \partial_n \)

\( 0 \)-cycles: \( \ker \partial_0 = C_0(X) \).

\( 1 \)-cycles: \( \partial_1(c_1 e_1 + c_2 e_2 + c_3 e_3) = \) \( c_1 (x_1 - x_0) + c_2 (x_1 - x_0) + c_3 (x_1 - x_0) \) \( = (c_1 + c_2 + c_3) x_1 - (c_1 + c_2 + c_3) x_0 \) \( \Rightarrow \ker \partial_1 = \) those with \( c_1 + c_2 + c_3 = 0 \).

\( H_n(X) = \frac{\ker \partial_n}{\text{im} \partial_{n+1}} \)

\( H_0(X) = \frac{C_0(X)}{C(x_1 - x_0)} = \frac{\mathbb{Z}^2}{(1, -1)} = \mathbb{Z} \)

\( H_1(X) = \frac{\ker \partial_1}{\text{im} \partial_2} = \frac{\ker \partial_1 = \\{ \text{those with } c_1 + c_2 + c_3 = 0 \}}{\text{with basis } e_1 - e_2, \quad e_2 - e_3} = \mathbb{Z}^2 \)
More complicated:

\[ C_2(X) = \mathbb{Z} \text{ gen by } d_1, \]
\[ \partial_2(d_1) = e_2 - e_1, \]
\[ H_1(X) = \ker \partial_1 / \text{im } \partial_2 = \mathbb{Z} \text{ gen by } e_2 - e_3 \]
\[ H_2(X) = 0 \text{ as } \ker \partial_2 = 0. \]

Finally:

\[ H_0 = \mathbb{Z}, \]
\[ H_1 = \mathbb{Z}, \]
\[ H_2 = \mathbb{Z} \text{ gen by } d_1 - d_2, \]
\[ H_n = 0 \text{ for } n > 2. \]