Last time: \((\tilde{X}, \tilde{x}_0) \to (X, x_0)\) a path conn. regular cover

\[ \tau : \pi_1 (X, x_0) \longrightarrow \text{group of covering transformations} \]

\[ \alpha \longrightarrow \text{covering trans taking } \]

\[ x_0 \text{ to } \tilde{x}(1) \text{ where } \alpha \text{ is the lift of } \alpha \text{ starting at } \tilde{x}_0. \]

**Thm:** \(\tau\) is onto and \(\ker \tau = \mathbb{Z} \cdot \pi_1 (\tilde{X}, \tilde{x}_0)\), i.e.

\[ \pi_1 X / \pi_1 \tilde{X} = \mathbb{Z} (\tilde{x}). \]

**Note:** \(\tau\) gives a map

\[ \pi_1 (X, x_0) \longrightarrow \text{Sym} \left( p^{-1}(x_0) \right) \]

\[ \alpha \longrightarrow \text{action of } \tau \text{ on } p^{-1}(x_0). \]
Also have the lifting permutation

\[ \tilde{\Pi}(X, x_0) \rightarrow \text{Sgm}(\tilde{p}^{-1}(x_0)) \]

\[ x \rightarrow L_\alpha \]

where \[ L_\alpha(\tilde{x}) = \tilde{x}(0) \] where \( \tilde{x} \) is the lift of \( \alpha \) ending at \( \tilde{x} \).

Important note: These are not related.

\[ \text{Ex: } \tilde{X} = \text{Univ. Cover} \]

\[ \text{Gar } \infty \]

Vertices:
Sets of Free Gp(\( a, b \))

Edges: between \( w \) and \( wa \)
\( w \) and \( wb \)

Example of a Cayley Graph.
Action of \( L_a \): Moves any vertex one unit to the left. In particular, any site moves at most one unit.

Action of \( T_a \):

is a "translation" along this axis.

In particular, \( d(w, T_a(w)) \) can be arbi. large.
Thm: \( \tilde{X} \) the univ. cover of \( X \). If \( H \leq \pi_1(X) \), then \( \tilde{X}/H \) is the cover over to \( H \).

**Group action:**

An action of a group \( G \) on a space \( X \) is a map \( G \times X \rightarrow X \) sat. the usual rules, and act on the second input.

**Ex:** \( G = \mathbb{Z}^2 = \langle a, b \rangle \) acts on \( \mathbb{R}^2 \)

via:

- \( a \cdot (x, y) = (x+1, y+1) \)
- \( b \cdot (x, y) = (x-1, y+2) \)

Diagram:

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A \  \ \ \  B \  \ \ \  C
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With arrows indicating the action of elements on points in \( \mathbb{R}^2 \).
Suppose $G$ acts on $X$. Give $X/G$ the quotient topology. When is $X \to X/G$ a covering map?

Yes: Above example

No: $\mathbb{Q}$ acts on $\mathbb{R}$ via translation: $q \cdot x = x + q$

Consider $\mathbb{R}/\mathbb{Q}$, it has the indiscrete topology; the only open sets are $\emptyset$ and the whole space.

 Pf: By def, if $U \subseteq \mathbb{R}/\mathbb{Q}$ is open then
\(p^{-1}(U)\) is open in \(\mathbb{R}\). So it contains an open interval, but it's also \(\mathbb{Q}\)-invariant. 
\[\Rightarrow p^{-1}(U) = \mathbb{R} \Rightarrow U = \mathbb{R}/\mathbb{Q}.\]

**Fact:** \(X \rightarrow X/\mathbb{G}\) is a covering map if:

1. \(G\) acts freely: i.e. \(g \cdot x = x \Rightarrow g = \text{id}.\)
2. Orbits don't accumulate: \(\forall x \in X, \exists a \text{ nbhd } U \text{ of } x \text{ s.t. all } g \cdot U \text{ for } g \in G\) are disjoint. That is \(g_1 \cdot U \cap g_2 \cdot U \neq \emptyset \Rightarrow g_1 = g_2.\)

\[\mathbb{C}_\mathbb{R} = X = \mathbb{H}^2 = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}\]
\[\mathbb{C} = \langle (1, 2), (1, 0) \rangle\]
\[X/\mathbb{C} = \triangle\]