Lecture 28: Equality of homologies

X a Δ-complex

Simplicial Homology: \( H^n_\Delta(X) \) — easy to compute.

Singular Homology: \( H_n(X) \) — seems to depend on cellulation.

Thm: \( H^n_\Delta(X) \cong H_n(X) \) — clearly invariant.

Lemma: \( H_n(\Delta^n, \partial \Delta^n) \cong \mathbb{Z} \) is generated by \( i_n = id_{\Delta^n} \).

Pf: Induction on \( n \). Clear for \( n = 0 \), i.e. (pt, φ).

\( \Lambda = \) All faces of \( \Delta^{n+1} \) except the 1st.

By the long exact seq of \( (\Lambda, \partial \Delta^{n+1}, \Delta^{n+1}) \):

\[
\begin{align*}
\rightarrow H_{n+1}(\Lambda, \partial \Delta^{n+1}) & \rightarrow H_{n+1}(\partial \Delta^{n+1}, \Lambda) \\
\cong 0 & \rightarrow H_n(\partial \Delta^{n+1}, \Lambda) \\
\ni & \rightarrow H_n(\Delta^{n+1}, \Lambda) \\
\cong \mathbb{Z} & \text{generators by } i_n
\end{align*}
\]
Claim: \( \partial [i_{n+1}] = i_* [i_n] \)

On the chain level,

\[
i_{n+1} \xrightarrow{\partial} \sum_{k=0}^{n+1} (-1)^k i_{n+1} \big|_{k^{th\text{ face}}} = i_n
\]

\( C_{n+1}(\Delta^{n+1}, \partial \Delta^{n+1}) \to C_n(\partial \Delta^{n+1}, \Lambda) \)

So \( H_{n+1}(\Delta^{n+1}, \partial \Delta^{n+1}) \) is gen by \( i_{n+1} \)

\[\textbf{Lemma:} \ x_\alpha \in X_\alpha \text{ s.t. } (X_\alpha, x_\alpha) \text{ is a good pair.} \]

If \( Y = \bigvee X_\alpha \) and \( i_\alpha : X_\alpha \to Y \) are the inclusions, then \( \bigoplus_{\alpha} i_\alpha : \bigoplus_{\alpha} \tilde{H}_n(X_\alpha) \to \tilde{H}_n(Y) \) is an isomorphism.

\[\textbf{Proof:} \quad \begin{array}{c}
\includegraphics{diagram}\end{array} \quad \text{By excision} \quad \begin{array}{c}
H_n\left( \bigvee_{\alpha} X_\alpha, \bigvee \{x_\alpha^3\} \right) \xrightarrow{q^*} \tilde{H}_n(Y)
\end{array}
\]

\( \bigoplus_{\alpha} H_n(X_\alpha, \{x_\alpha^3\}) \quad \text{induced by inclusion} \)
Then we have a chain map 

\[ C_n(X) \longrightarrow C_n(X) \]

generated by \( \sigma_\alpha : \Delta^n \rightarrow X \).

**Thm:** \( H^n_\Delta(X) \rightarrow H^n(X) \) is an isomorphism.

**Pf:** Assume \( X \) is finite dim'd [full case in Hatcher.]

Inductively, show \( H^n_\Delta(X^k) \cong H^n_\Delta(X^k) \)

Base case \( X^0 = \{ \text{pts} \} \) is clear.

Assume true for \( k \). Have

\[
H_{n+1}^\Delta(X^{k+1}, X^k) \rightarrow H_n^\Delta(X^k) \\
\downarrow \cong \downarrow \cong \downarrow \cong \downarrow \cong \downarrow \cong \\
H_{n+1}^\Delta(X^{k+1}, X^k) \rightarrow H_n^\Delta(X_{k+1}^k) \\
\downarrow \cong \downarrow \cong \downarrow \cong \downarrow \cong \\
H_{n+1}^\Delta(X^{k+1}, X^k) \rightarrow H_n^\Delta(X_{k+1}^k) \\
\downarrow \cong \downarrow \cong \downarrow \cong \downarrow \cong \\
H_{n+1}^\Delta(X^{k+1}, X^k) \rightarrow H_n^\Delta(X_{k+1}^k)
\]

Now \( X^{k+1}/X^k = \bigvee_{\alpha} S^{k+1} \) with one sphere for each \( k+1 \) cell.

And \( \tilde{H}_n(S^{k+1}) = 0 \) for \( n \neq k+1 \),

\[ \tilde{H}_n(S^{k+1}) = \begin{cases} \mathbb{Z} & n = k+1 \\ 0 & \text{otherwise} \end{cases} \]
So \( H_n(X^{k+1}, X^k) = \left\{ \begin{array}{ll} \bigoplus Z & \text{sum over the } k+1 \text{ cells} \\ 0 & \text{otherwise} \end{array} \right. \)

Same is true for \( H_n^\Delta(X^{k+1}, X^k) \) as the only non-zero chain group is \( C_n^\Delta(X^{k+1}, X^k) = \bigoplus(Z, \text{gen by } \varepsilon \alpha) \).

Moreover,

\[
H_n^\Delta(X^{k+1}, X^k) = \bigoplus \alpha \left( Z, \text{gen by } \varepsilon \alpha \right)
\]

\[
\Rightarrow H_n(X^{k+1}, X^k) \cong \bigoplus \alpha \left( Z, \text{gen by } \varepsilon \alpha \right) = \bigoplus Z
\]

is an isom by the lemmas.

Everything now follows from

**Five Lemma:**

\[
\begin{array}{cccccc}
A & \rightarrow & B & \rightarrow & C & \rightarrow & D & \rightarrow & E \\
\downarrow {\alpha} & & \downarrow {\beta} & & \downarrow {\gamma} & & \downarrow {\delta} & & \downarrow {\varepsilon} \\
A' & \rightarrow & B' & \rightarrow & C' & \rightarrow & D' & \rightarrow & E'
\end{array}
\]

If the top and bottom are exact, and \( \alpha, \beta, \delta, \varepsilon \) are \( \cong \) then \( Y \) is also an \( \cong \).

**Of:**

Diagram chase.