Goal: Good Pair: A closed \( A \subseteq X \) has a nbhd \( V \) which df
restricts to it.

Then: \((X,A)\) a good pair. Then the quotient map

\[ g: (X,A) \to (X/A, A/A) \]

induces an isom.

\[ H_n(X,A) \overset{g^*}{\to} H_n(X/A, A/A) \cong \tilde{H}_n(X/A) \quad \text{for all } n. \]

[Combines to give the long exact seq of the pair.]

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**Excision:** \( Z \subseteq A \subseteq X \), with \( \overline{Z} \subseteq \text{int}(A) \).

Then the map \( \sqrt{\text{inclusion}} \)

\[ H_n(X \setminus \overline{Z}, A \setminus \overline{Z}) \overset{i_*}{\to} H_n(X, A) \]

is an isomorphism.

[Motivate, jump up and down about, etc.]

Pf of Thm from Excision: In general, \( H_n(Y, pt) \cong \tilde{H}_n(Y) \)
since

\[ \to H_n(pt) \to H_n(Y) \cong H_n(Y, pt) \to H_{n-1}(pt) \to \]

---

0
Claim: $H_n(X, A) \xrightarrow{i_*} H_n(X, V)$

Proof: Long exact sequence of the triple

$0 \rightarrow H_n(V, A) \rightarrow H_n(X, A) \xrightarrow{i_*} H_n(X, V) \xrightarrow{q_*} H_{n-1}(V, A) \rightarrow 0$

comes from the short exact sequence

$0 \rightarrow C_*(V, A) \rightarrow C_*(X, A) \rightarrow C_*(X, V) \rightarrow 0$

of chain complexes.

By long exact seq of the triple

$H_n(X, A) \xrightarrow{i_*} H_n(X, V) \xrightarrow{\cong} H_n(X \setminus A, V \setminus A)$

by Excision

$H_n(X/A, A/A) \rightarrow H_n(X/A, V/A) \xrightarrow{\cong} H_n(X/A \setminus A/A, V/A \setminus A/A)$

by Excision

$g_* \xrightarrow{\cong} g_{|A}$ is a homeo
Commuting of the two squares forces leftmost \( q^* \) to be an \( \cong \).

[Because of missed classes, won't prove Excision in detail.]

**Setup:** \( U = \{ U_i \} \) with \( \text{int}(U_i) = X \).

Let \( C_n^U(X) \leq C_n(X) \) given \( e: \Delta^n \to X \) with image \( \subseteq \text{some } U_i \).

A subcomplex, so have \( H_n^U(X) \).

For excision, take \( U = \{ A, B = X - Z \} \)

**Reason:** \[ C_n(X-Z, A-Z) \xrightarrow{i} C_n(X, A) \]

Want a map back, but only maps sense on \( C_n^U(X) \).
Prop: \( \iota: C_*^u(X) \longrightarrow C_*(X) \) is a chain homotopy equivalence, \( \exists \rho: C_*(X) \longrightarrow C_*(X) \) s.t. \( \rho \circ \iota \) and \( \iota \circ \rho \) are chain homotopic to the identity. In particular, \( H_n^u(X) \longrightarrow H_n(X) \) is an isom.

Constructing \( \rho \):

Barycentric Subdivision:

\[ \delta = [v_0, v_1, \ldots, v_n] \text{ an } n \text{-simplex} \]

Barycenter: \( \frac{1}{n+1} \sum_i v_i = \tau \)

Subdivision \( \delta \rho \):

\[ \delta \rho: C_n(X) \longrightarrow C_n(X) \]

\[ \delta \rho \longrightarrow \sum \pm \delta | \tau \]

\( \tau \) in barycentric subdivision of \( \Delta^n \)

A chain map chain hom. to the ident.
Rough idea: \( p = S^n \), but not quite as the number of times we need to subdivide depends on \( p \). See Hatcher for details.

Next time: \( H_n^\Delta(X) \cong H_n(X) \).

Then onto applications!