Lecture 26: Relative Homology

**Goal:** \[ \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A) \to \]

**Relative Homology:** (Stand-in for \( H_n(X/A) \))

**New complex:** \[ C_n(X, A) = C_n(X) / C_n(A) \text{ with induced } \partial \text{-map.} \]

**Relative Homology:** \( H_n(X, A) \) [= the homology of this chain complex]

**Relative cycles:** \( c \in C_n(X) \text{ with } \partial c \in C_{n-1}(A) \)

\( C = 0 \text{ in } H_n(X, A) \iff \text{relative boundary: } \)

\[ c = \partial d + a \text{ with } de \in C_{n+1}(X), \ a \in C_n(A) \]

**Ex:** \[ X = I, \ A = \partial I \]

Compute using \( \Delta \)-complex homology.
\[ C_2(X,A) \rightarrow C_1(X,A) \rightarrow C_0(X,A) \rightarrow 0 \]
\[ \mathbb{Z} = \langle \delta_1 \rangle \rightarrow 0 \]
\[ \Rightarrow H_n^{\Delta}(I, J) = \begin{cases} \mathbb{Z} & \text{n} = 1 \\ 0 & \text{otherwise} \end{cases} \]
\[ \cong H_n(I/\partial I \cong S) \]

Have a long exact seq involving \( H_n(X,A) \), as an instance of a general setup:

\[ 0 \rightarrow A_n \overset{i}{\rightarrow} B_n \overset{j}{\rightarrow} C_n \rightarrow 0 \quad \text{is exact at } A_n. \]

Thm: A short exact seq of chain complexes

\[ 0 \rightarrow A_* \overset{i}{\rightarrow} B_* \overset{j}{\rightarrow} C_* \rightarrow 0 \]

where \( i \) and \( j \) are chain maps, gives rise to a long exact sequence:

\[ \cdots \rightarrow H_n(A) \overset{i_*}{\rightarrow} H_n(B) \overset{j_*}{\rightarrow} H_n(C) \overset{\partial}{\rightarrow} H_{n-1}(A) \overset{i_*}{\rightarrow} H_{n-1}(B) \overset{j_*}{\rightarrow} \cdots \]

\[ \cong \]

\[ A_* = C_*(A), \quad B_* = C_*(X), \quad C_* = C_*(X,A). \]

\[ \cdots \rightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X,A) \rightarrow H_{n-1}(A) \rightarrow H_{n-1}(X) \rightarrow \cdots \]
Def of $\partial: H_n(C) \to H_{n-1}(A)$:

\[
\begin{array}{cccc}
& 0 & \downarrow & 0 & \downarrow & 0 \\
\downarrow & & & & & \\
A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} & \rightarrow \\
\downarrow & i & \downarrow & i & \downarrow & \\
B_{n+1} & \rightarrow & B_n & \rightarrow & B_{n-1} & \\
\downarrow & j & \uparrow & \downarrow & c & \downarrow \\
C_{n+1} & \rightarrow & C_n & \rightarrow & C_{n-1} & \\
\downarrow & & & & \downarrow & \\
0 & & 0 & & 0 & 
\end{array}
\]

Start with $[c] \in H_n(C)$. Pick $b \in B_n$ with $j(b) = c$. Since $j(\partial b) = \partial(jb) = \partial c = 0$, $\exists a \in A_{n-1}$ with $i(a) = \partial b$.

Let $\partial[c] = [a]$, which is in $H_{n-1}(A)$ since

$\partial a = 0 \iff 0 = i(\partial a) = \partial(i(a)) = \partial^2 b = 0$.

Why is this well-defined?
• a only dep on b, as i is 1-1

• clif b' also has j(b') = c, \( \exists d \in A_n \) with \( i(d) = b' - b \)

Thus \( d \rightarrow a' - a \) since i is 1-1.

\( b' - b \rightarrow \partial b' - \partial b \)  \( \therefore [a'] = [a] \) in \( H_{n-1}(A) \).

• clif we replace c with \( c + d \) where \( d \in C_{n+1} \),
then pick \( b'' \in B_{n+1} \) with \( j(b'') = d \). Then

\( b'' \rightarrow b'' + \partial b'' \) we can take \( b' = b + \partial b'' \) and so \( \partial b = \partial b' \) and
so this doesn't change anything.

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\textbf{Proof:}

\underline{Exactness at \( H_n(B) \):}\ \( \text{im } i_* \subseteq \ker j_* \) \( j \circ i = 0 \) at chain level

\( \ker j_* \subseteq \text{im } i_* \) \( b \in B_n \) a cycle with \( j_*(\{b\}) = 0 \),
i.e. \( j(b) = \partial c \). Pick \( b' \in B_{n+1} \) with \( j(b') = c \).
Then $b - \partial b'$ is in the kernel of $j$, so pick $a \in A_n$ with $i(a) = b - \partial b'$.

Now $[a] \in \text{H}_n(A)$ since

$$i(\partial a) = \partial (i(a)) = \partial b - \partial^2 b' = 0, \text{ and moreover}$$

$$i_x([a]) = [i(a)] = [b - \partial b'] = [b].$$

Exactness at other locations are similar.

**Geometric clustep of $d$:**