Lecture 2: The invariants of algebraic topology

Algebraic Topology: $X \mapsto F(X)$ alg. object, e.g. a group, top space [depends only on the homeomorphism type]

Main invariants:

Fundamental Group [Ch 1 of Hatcher]

$\Pi_1 X = \text{"the set of loops in } X \text{ up to continuous deformation."}$

Ex: $X = \mathbb{R}^2 \setminus B_1(0)$

Three loops that are all the same in $\Pi_1 X$:

A loop that's different

Formally, a loop is a (continuous) map from

$(S^1 = \text{circle} = \mathbb{C}) \rightarrow X$

In this case, there's a $\mathbb{Z}$'s worth of loops. (c.f. winding # in complex anal.)
Example: $Y = \mathbb{R}^2 \setminus (B_1((2,0)) \cup B_1((-2,0)))$

Note: $\pi_1 X$ and $\pi_1 Y$ are both countably infinite. [To tell them apart, need to make $\pi_1$ into a group.]

Fix a point $y_0$ in $Y$. [the basepoint.] and just consider loops that start at $y_0$.

Group operation: concatenation.

$\beta \ast \alpha \neq \alpha \ast \beta$. 
For $X$, notice that if

\[ \alpha * \beta \]

then $\alpha \star \beta$

That is, $\beta = \alpha^{-1}$. Turns out that

\[ \pi_1 X = \{ \alpha^n \mid n \in \mathbb{Z} \} \xrightarrow{\sim} (\mathbb{Z}, n) \]

loop $\longrightarrow$ winding number

In contrast, $\pi_1 Y$ is not commutative (see last page), so $\pi_1 Y \neq \pi_1 X$ and $Y \neq X$.

Higher Homotopy Groups $[\text{Math 526; Ch 4 of Hatcher}]

\[ S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \} \]

$\bigcirc$, \, $\bigcirc$, $S^3 = \mathbb{R}^3 \cup \{0\}$, etc...

\[ \pi_n (X) = \{ S^n \rightarrow X \} \xrightarrow{\text{up to deformation}} \]

For $n > 1$, this is an abelian group; very powerful.
capturing a huge amount of info about $X$. Really hard to compute, e.g. $\Pi_n(S^2 = \emptyset)$ are not all known.

**Homology:** [Ch 2 of Hatcher]

$H_n(X) = \text{ `n-dim' things w/o boundary}$

$\text{an abelian group}$

an all of $H_1(X)$ is countable

= 0 in $H_1(X)$

[Easy to compute, but still very useful.]

[Ch 3 of Hatcher]

**Cohomology:**

$H^n(X)$ - "dual" notion to homology.

When $X$ is a smooth manifold, can be defined using differential forms.

The $H^n(X) \times H^m(X) \rightarrow H^{n+m}(X)$, making

$\bigoplus H^n(X)$ into an algebra.

We will cover

(bo) homology - from a topological viewpoint, but they are an important tool in many areas of math.