Topological Spaces: set of points w/ notion of nearness.

$\mathbb{R}^n$, $\mathbb{S}^2$, $\mathbb{C}$, $\mathbb{S}^3$ ...

circle, sphere, configuration space of a robot arm...

Metric Space: $(X, d)$

$\mathbb{R}^2$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

From this define

1. Convergence of a sequence
2. Continuity of functions.
3. Closed and open sets.

using open balls

$B_\varepsilon(x) = \{ y \in X \mid d(x, y) < \varepsilon \}$
$U \subseteq X$ is open if $\forall x \in X \ \exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Some other metrics on $\mathbb{R}^2$ define the same open sets, e.g.,

$$d'(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

In geometry, the focus is on the metric; in topology, it's on the open sets.

**Def:** A topological space is a set $X$ with a collection $U$ of subsets sat:

1. $\emptyset$ and $X$ are in $U$.
2. $U_1, U_2 \in U \Rightarrow U_1 \cap U_2 \in U$.
3. $U_\alpha \in U \Rightarrow \bigcup U_\alpha \in U$.

**Ex:** $X$ a metric sp. $U$ = usual open sets.

[Most of the examples we'll focus on in this class are actually metric spaces.]

**Def:** $f : X \to Y$ is continuous if $\forall$ open $U \subseteq Y$, $f^{-1}(U)$ is open in $X$. 

(Write some diagrams or equations here if needed.)
Top spaces $X$ and $Y$ are homeomorphic if there is a bijection $f: X \rightarrow Y$ where $f$ and $f^{-1}$ are both continuous.

**Problem:** Given $X, Y$ are they homeomorphic?

**Yes:** Give homeo: \[
\begin{array}{c}
\text{\includegraphics{coffee}} \\
\rightarrow \\
\text{\includegraphics{tea}} \\
\rightarrow \\
\text{\includegraphics{cup}} \\
\end{array}
\]

**No:** Need a property to distinguish them.

**Easy:** $(0,1) \neq [0,1]$ Compactness

$\mathbb{R} \neq \mathbb{R}^2$ Connectedness of space $\setminus$ pt.

**Harder:** $\mathbb{R}^2 \neq \mathbb{R}^3$, $\mathbb{S}^2 \neq \mathbb{S}^3$.

[Need additional prop/rivariant. Am simp source of such is:

Algebraic Topology:

$X \xrightarrow{\sim} F(X)$

[topsp]

[depends only on the homeo type of $X$]

Also $X \xrightarrow{\text{cont}} Y$ gives $F(X) \xrightarrow{f_*} F(Y)$ Respect the alg str.
Brouwer Fixed Point Thm: \( D^n = \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \} \)

If \( f: D^n \to D^n \) is cont, then \( \exists x \in D^n \) with \( f(x) = x \).

Geometric Topology: Study of spaces that are locally nice, e.g., manifolds.

Course in a nutshell:

Invariants of algebraic topology as applied to the examples of geometric topology.

Discuss syllabus, etc.