
3. Prove Lemma 3.2.

4. Let $S$ be a set with a partial order relation $\leq$. A standard technique in combinatorics is to associate with $S$ the abstract complex $\mathcal{S}$ whose vertices are the elements of $S$ and whose simplices are the finite simply-ordered subsets of $S$. Suppose one is given the partial order on $\{a_1, \ldots, a_n\}$ generated by the following relations:

$$a_1 \leq a_3 \leq a_5 \leq a_6; \quad a_1 \leq a_5 \leq a_7;$$

$$a_2 \leq a_4 \leq a_5; \quad a_2 \leq a_5.$$

Describe a geometric realization of $\mathcal{S}$.

§4. REVIEW OF ABELIAN GROUPS

In this section, we review some results from algebra that we shall be using—specifically, facts about abelian groups.

We write abelian groups additively. Then $0$ denotes the neutral element, and $-g$ denotes the additive inverse of $g$. If $n$ is a positive integer, then $ng$ denotes the $n$-fold sum $g + \cdots + g$, and $(-n)g$ denotes $n(-g)$.

We denote the group of integers by $\mathbb{Z}$, the rationals by $\mathbb{Q}$, and the complex numbers by $\mathbb{C}$.

**Homomorphisms**

If $f : G \to H$ is a homomorphism, the kernel of $f$ is the subgroup $f^{-1}(0)$ of $G$, the image of $f$ is the subgroup $f(G)$ of $H$, and the cokernel of $f$ is the quotient group $H/f(G)$. We denote these groups by $\ker f$ and $\text{im } f$ and $\text{cok } f$, respectively. The map $f$ is a monomorphism if and only if the kernel of $f$ vanishes (i.e.,
equals the trivial group). And $f$ is an epimorphism if and only if the cokernel of $f$ vanishes; in this case, $f$ induces an isomorphism $G/\ker f \cong H$.

**Free abelian groups**

An abelian group $G$ is **free** if it has a **basis**—that is, if there is a family $\{g_a\}_{a \in A}$ of elements of $G$ such that each $g \in G$ can be written uniquely as a finite sum

$$g = \sum n_ag_a,$$

with $n_a$ an integer. Uniqueness implies that each element $g_a$ has infinite order; that is, $g_a$ generates an infinite cyclic subgroup of $G$.

More generally, if each $g \in G$ can be written as a finite sum $g = \sum n_ag_a$, but not necessarily uniquely, then we say that the family $\{g_a\}$ **generates** $G$. In particular, if the set $\{g_a\}$ is finite, we say that $G$ is **finitely generated**.

If $G$ is free and has a basis consisting of $n$ elements, say $g_1, \ldots, g_n$, then it is easy to see that every basis for $G$ consists of precisely $n$ elements. For the group $G/2G$ consists of all cosets of the form

$$(\sum \epsilon_ig_i) + 2G,$$

where $\epsilon_i = 0$ or $1$; this fact implies that the group $G/2G$ consists of precisely $2^n$ elements. The number of elements in a basis for $G$ is called the **rank** of $G$.

It is true more generally that if $G$ has an infinite basis, any two bases for $G$ have the same cardinality. We shall not use this fact.

A crucial property of free abelian groups is the following: If $G$ has a basis $\{g_a\}$, then any function $f$ from the set $\{g_a\}$ to an abelian group $H$ extends uniquely to a homomorphism of $G$ into $H$.

One specific way of constructing free abelian groups is the following: Given a set $S$, we define the **free abelian group** $G$ **generated by** $S$ to be the set of all functions $\phi : S \to \mathbb{Z}$ such that $\phi(x) \neq 0$ for only finitely many values of $x$; we add two such functions by adding their values. Given $x \in S$, there is a characteristic function $\phi_x$ for $x$, defined by setting

$$\phi_x(y) = \begin{cases} 0 & \text{if } y \neq x, \\ 1 & \text{if } y = x. \end{cases}$$

The functions $\{\phi_x \mid x \in S\}$ form a basis for $G$, for each function $\phi \in G$ can be written uniquely as a finite sum

$$\phi = \sum n_x\phi_x,$$

where $n_x = \phi(x)$ and the summation extends over all $x$ for which $\phi(x) \neq 0$. We often abuse notation and identify the element $x \in S$ with its characteristic function $\phi_x$. With this notation, the general element of $G$ can be written uniquely as a finite "formal linear combination"

$$\phi = \sum n_x x_a$$

of the elements of the set $S$. 

If \( G \) is an abelian group, an element \( g \) of \( G \) has finite order if \( ng = 0 \) for some positive integer \( n \). The set of all elements of finite order in \( G \) is a subgroup \( T \) of \( G \), called the torsion subgroup. If \( T \) vanishes, we say \( G \) is torsion-free. A free abelian group is necessarily torsion-free, but not conversely.

If \( T \) consists of only finitely many elements, then the number of elements in \( T \) is called the order of \( T \). If \( T \) has finite order, then each element of \( T \) has finite order; but not conversely.

**Internal direct sums**

Suppose \( G \) is an abelian group, and suppose \( \{G_\alpha\} \) is a collection of subgroups of \( G \), indexed bijectively by some index set \( J \). Suppose that each \( g \) in \( G \) can be written uniquely as a finite sum \( g = \sum g_\alpha \), where \( g_\alpha \in G_\alpha \) for each \( \alpha \). Then \( G \) is said to be the internal direct sum of the groups \( G_\alpha \), and we write

\[
G = \bigoplus_{\alpha \in J} G_\alpha.
\]

If the collection \( \{G_\alpha\} \) is finite, say \( \{G_\alpha\} = \{G_1, \ldots, G_n\} \), we also write this direct sum in the form \( G = G_1 \oplus \cdots \oplus G_n \).

If each \( g \) in \( G \) can be written as a finite sum \( g = \sum g_\alpha \), but not necessarily uniquely, we say simply that \( G \) is the sum of the groups \( \{G_\alpha\} \), and we write \( G = \sum G_\alpha \), or, in the finite case, \( G = G_1 + \cdots + G_n \). In this situation, we also say that the groups \( \{G_\alpha\} \) generate \( G \).

If \( G = \sum G_\alpha \), then this sum is direct if and only if the equation \( 0 = \sum g_\alpha \) implies that \( g_\alpha = 0 \) for each \( \alpha \). This in turn occurs if and only if for each fixed index \( \alpha_0 \), one has

\[
G_{\alpha_0} \cap \left( \sum_{\alpha \neq \alpha_0} G_\alpha \right) = \{0\}.
\]

In particular, if \( G = G_1 + G_2 \), then this sum is direct if and only if \( G_1 \cap G_2 = \{0\} \).

The resemblance to free abelian groups is strong. Indeed, if \( G \) is free with basis \( \{g_\alpha\} \), then \( G \) is the direct sum of the subgroups \( \{G_\alpha\} \), where \( G_\alpha \) is the infinite cyclic group generated by \( g_\alpha \). Conversely, if \( G \) is a direct sum of infinite cyclic subgroups, then \( G \) is a free abelian group.

If \( G \) is the direct sum of subgroups \( \{G_\alpha\} \), and if for each \( \alpha \), one has a homomorphism \( f_\alpha \) of \( G_\alpha \) into the abelian group \( H \), the homomorphisms \( \{f_\alpha\} \) extend uniquely to a homomorphism of \( G \) into \( H \).

Here is a useful criterion for showing \( G \) is a direct sum:

**Lemma 4.1.** Let \( G \) be an abelian group. If \( G \) is the direct sum of the subgroups \( \{G_\alpha\} \), then there are homomorphisms

\[
j_\beta : G_\beta \to G \quad \text{and} \quad \pi_\beta : G \to G_\beta
\]

such that \( \pi_\beta \circ j_\alpha \) is the zero homomorphism if \( \alpha \neq \beta \), and the identity homomorphism if \( \alpha = \beta \).
Conversely, suppose \( \{G_\alpha\} \) is a family of abelian groups, and there are homomorphisms \( j_\beta \) and \( \pi_\beta \) as above. Then \( j_\beta \) is a monomorphism. Furthermore, if the groups \( j_\alpha(G_\alpha) \) generate \( G \), then \( G \) is their direct sum.

Proof. Suppose \( G = \bigoplus G_\alpha \). We define \( j_\beta \) to be the inclusion homomorphism. To define \( \pi_\beta \), write \( g = \sum g_\alpha \), where \( g_\alpha \in G_\alpha \) for each \( \alpha \); and let \( \pi_\beta(g) = g_\beta \). Uniqueness of the representation of \( g \) shows \( \pi_\beta \) is a well-defined homomorphism.

Consider the converse. Because \( \pi_\alpha \circ j_\alpha \) is the identity, \( j_\alpha \) is injective (and \( \pi_\alpha \) is surjective). If the groups \( j_\alpha(G_\alpha) \) generate \( G \), every element of \( G \) can be written as a finite sum \( \sum j_\alpha(g_\alpha) \), by hypothesis. To show this representation is unique, suppose

\[
\sum j_\alpha(g_\alpha) = \sum j_\beta(g'_\beta).
\]

Applying \( \pi_\beta \), we see that \( g_\alpha = g'_\beta \). \( \square \)

Direct products and external direct sums

Let \( \{G_\alpha\}_{\alpha \in J} \) be an indexed family of abelian groups. Their direct product \( \prod_{\alpha \in J} G_\alpha \) is the group whose underlying set is the cartesian product of the sets \( G_\alpha \), and whose group operation is component-wise addition. Their external direct sum \( G \) is the subgroup of the direct product consisting of all tuples \( (g_\alpha)_{\alpha \in J} \) such that \( g_\alpha = 0_\alpha \) for all but finitely many values of \( \alpha \). (Here \( 0_\alpha \) is the zero element of \( G_\alpha \).) The group \( G \) is sometimes also called the "weak direct product" of the groups \( G_\alpha \).

The relation between internal and external direct sums is described as follows: Suppose \( G \) is the external direct sum of the groups \( \{G_\alpha\} \). Then for each \( \beta \), we define \( \pi_\beta : G \to G_\beta \) to be projection onto the \( \beta \)th factor. And we define \( j_\beta : G_\beta \to G \) by letting it carry the element \( g \in G_\beta \) to the tuple \( (g_\alpha)_{\alpha \in J} \), where \( g_\alpha = 0_\alpha \) for all \( \alpha \) different from \( \beta \), and \( g_\beta = g \). Then \( \pi_\beta \circ j_\alpha = 0 \) for \( \alpha \neq \beta \), and \( \pi_\alpha \circ j_\beta = 0 \). It follows that \( G \) equals the internal direct sum of the groups \( G_\alpha = j_\alpha(G_\alpha) \), where \( G_\alpha \) is isomorphic to \( G_\alpha \).

Thus the notions of internal and external direct sums are closely related. The difference is mainly one of notation. For this reason, we customarily use the notations

\[
G = G_1 \oplus \cdots \oplus G_n \quad \text{and} \quad G = \bigoplus G_\alpha
\]

to denote either internal or external direct sums, relying on the context to make clear which is meant (if indeed, it is important). With this notation, one can for instance express the fact that \( G \) is free abelian of rank 3 merely by writing \( G \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \).

If \( G_1 \) is a subgroup of \( G \), we say that \( G_1 \) is a direct summand in \( G \) if there is a subgroup \( G_2 \) of \( G \) such that \( G = G_1 \oplus G_2 \). In this case, if \( H_1 \) is a subgroup of \( G_i \), for \( i = 1, 2 \), then the sum \( H_1 + H_2 \) is direct, and furthermore,

\[
\frac{G}{H_1 \oplus H_2} \cong \frac{G_1}{H_1} \oplus \frac{G_2}{H_2}.
\]
In particular, if \( G = G_1 \oplus G_2 \), then \( \frac{G}{G_1} \cong G_2 \).

Of course, one can have \( \frac{G}{G_1} \cong G_2 \) without its following that \( G = G_1 \oplus G_2 \); that is, \( G_1 \) may be a subgroup of \( G \) without being a direct summand in \( G \). For instance, the subgroup \( n\mathbb{Z} \) of the integers is not a direct summand in \( \mathbb{Z} \), for that would mean that

\[
\mathbb{Z} \cong n\mathbb{Z} \oplus G_2
\]

for some subgroup \( G_2 \) of \( \mathbb{Z} \). But then \( G_2 \) is isomorphic to \( \mathbb{Z}/n\mathbb{Z} \), which is a group of finite order, while no subgroup of \( \mathbb{Z} \) has finite order.

Incidentally, we shall denote the group \( \mathbb{Z}/n\mathbb{Z} \) of integers modulo \( n \) simply by \( \mathbb{Z}/n \), in accordance with current usage.

The fundamental theorem of finitely generated abelian groups

There are actually two theorems that are important to us. The first is a theorem about subgroups of free abelian groups. We state it here, and give a proof in §11:

**Theorem 4.2.** Let \( F \) be a free abelian group. If \( R \) is a subgroup of \( F \), then \( R \) is also a free abelian group. If \( F \) has rank \( n \), then \( R \) has rank \( r \leq n \); furthermore, there is a basis \( e_1, \ldots, e_n \) for \( F \) and integers \( t_1, \ldots, t_k \) with \( t_i > 1 \) such that

1. \( t_i e_1, \ldots, t_k e_k, e_{k+1}, \ldots, e_n \) is a basis for \( R \).
2. \( t_1 | t_2 | \cdots | t_k \), that is, \( t_i \) divides \( t_{i+1} \) for all \( i \).

The integers \( t_1, \ldots, t_k \) are uniquely determined by \( F \) and \( R \), although the basis \( e_1, \ldots, e_n \) is not.

An immediate corollary of this theorem is the following:

**Theorem 4.3 (The fundamental theorem of finitely generated abelian groups).** Let \( G \) be a finitely generated abelian group. Let \( T \) be its torsion subgroup.

(a) There is a free abelian subgroup \( H \) of \( G \) having finite rank \( \beta \) such that \( G = H \oplus T \).

(b) There are finite cyclic groups \( T_1, \ldots, T_k \), where \( T_i \) has order \( t_i > 1 \), such that \( t_1 | t_2 | \cdots | t_k \) and

\[
T = T_1 \oplus \cdots \oplus T_k.
\]

(c) The numbers \( \beta \) and \( t_1, \ldots, t_k \) are uniquely determined by \( G \).

The number \( \beta \) is called the betti number of \( G \); the numbers \( t_1, \ldots, t_k \) are called the torsion coefficients of \( G \). Note that \( \beta \) is the rank of the free abelian group \( G/T \cong H \). The rank of the subgroup \( H \) and the orders of the subgroups \( T_i \) are uniquely determined, but the subgroups themselves are not.
Proof. Let $S$ be a finite set of generators $\{g_i\}$ for $G$; let $F$ be the free abelian group on the set $S$. The map carrying each $g_i$ to itself extends to a homomorphism carrying $F$ onto $G$. Let $R$ be the kernel of this homomorphism. Then $F/R \cong G$. Choose bases for $F$ and $R$ as in Theorem 4.2. Then

$$F = F_1 \oplus \ldots \oplus F_n,$$

where $F_i$ is infinite cyclic with generator $e_i$; and

$$R = t_1F_1 \oplus \ldots \oplus t_kF_k \oplus F_{k+1} \oplus \ldots \oplus F_r.$$

We compute the quotient group as follows:

$$F/R \cong (F_1/t_1F_1 \oplus \ldots \oplus F_k/t_kF_k) \oplus (F_{r+1} \oplus \ldots \oplus F_n).$$

Thus there is an isomorphism

$$f : G \rightarrow (\mathbb{Z}/t_1 \oplus \ldots \oplus \mathbb{Z}/t_k) \oplus (\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}).$$

The torsion subgroup $T$ of $G$ must be mapped to the subgroup $\mathbb{Z}/t_1 \oplus \ldots \oplus \mathbb{Z}/t_k$ by $f$, since any isomorphism preserves torsion subgroups. Parts (a) and (b) of the theorem follow. Part (c) is left to the exercises. \(\square\)

This theorem shows that any finitely generated abelian group $G$ can be written as a finite direct sum of cyclic groups; that is,

$$G \cong (\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}) \oplus \mathbb{Z}/t_1 \oplus \ldots \oplus \mathbb{Z}/t_k,$$

with $t_i > 1$ and $t_1 \mid t_2 \mid \ldots \mid t_k$. This representation is in some sense a "canonical form" for $G$. There is another such canonical form, derived as follows:

Recall first the fact that if $m$ and $n$ are relatively prime positive integers, then

$$\mathbb{Z}/m \oplus \mathbb{Z}/n \cong \mathbb{Z}/mn.$$

It follows that any finite cyclic group can be written as a direct sum of cyclic groups whose orders are powers of primes. Theorem 4.3 then implies that for any finitely generated group $G$,

$$G \cong (\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}) \oplus (\mathbb{Z}/a_1 \oplus \ldots \oplus \mathbb{Z}/a_i),$$

where each $a_i$ is a power of a prime. This is another canonical form for $G$, since the numbers $a_i$ are uniquely determined by $G$ (up to a rearrangement), as we shall see. The numbers $a_i$ are called the invariant factors of $G$.

**EXERCISES**

1. Show that if $G$ is a finitely generated abelian group, every subgroup of $G$ is finitely generated. (This result does not hold for non-abelian groups.)

2. (a) Show that if $G$ is free, then $G$ is torsion-free.
   (b) Show that if $G$ is finitely generated and torsion-free, then $G$ is free.
(c) Show that the additive group of rationals $\mathbb{Q}$ is torsion-free but not free. 
[Hint: If $\{g_{n}\}$ is a basis for $\mathbb{Q}$, let $\beta$ be fixed and express $g_{\alpha}/2$ in terms of 
this basis.]

3. (a) Show that if $m$ and $n$ are relatively prime, then $\mathbb{Z}/m \oplus \mathbb{Z}/n$ is cyclic of 
order $mn$.

(b) If $G \simeq \mathbb{Z}/18 \oplus \mathbb{Z}/36$, express $G$ as a direct sum of cyclic groups of prime 
power order.

(c) If $G \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9$, find the torsion 
coefficients of $G$.

(d) If $G \simeq \mathbb{Z}/15 \oplus \mathbb{Z}/20 \oplus \mathbb{Z}/18$, find the invariant factors and the torsion 
coefficients of $G$.

4. (a) Let $p$ be prime; let $b_{1}, \ldots, b_{k}$ be non-negative integers. Show that if

$$G \simeq (\mathbb{Z}/p)^{b_{1}} \oplus (\mathbb{Z}/p^{2})^{b_{2}} \oplus \ldots \oplus (\mathbb{Z}/p^{k})^{b_{k}},$$

then the integers $b_{i}$ are uniquely determined by $G$. [Hint: Consider the 
kernel of the homomorphism $f_{i}: G \rightarrow G$ that is multiplication by $p^{i}$. Show 
that $f_{i}$ and $f_{j}$ determine $b_{i}$. Proceed similarly.]

(b) Let $p_{1}, \ldots, p_{N}$ be a sequence of distinct primes. Generalize (a) to a finite 
direct sum of terms of the form $(\mathbb{Z}/p^{i})^{b_{i}}$, where $b_{i} \geq 0$.

(c) Verify (c) of Theorem 4.3. That is, show that the betti number, invariant 
factors, and torsion coefficients of a finitely generated abelian group $G$ are 
uniquely determined by $G$.

(d) Show that the numbers $t_{i}$ appearing in the conclusion of Theorem 4.2 are 
uniquely determined by $F$ and $R$.

§5. HOMOLOGY GROUPS

Now we are ready to define the homology groups. First we must discuss the 
notion of “orientation.”

Definition. Let $\sigma$ be a simplex (either geometric or abstract). Define two 
orderings of its vertex set to be equivalent if they differ from one another by an 
even permutation. If dim $\sigma > 0$, the orderings of the vertices of $\sigma$ then fall into 
two equivalence classes. Each of these classes is called an orientation of $\sigma$. (If $\sigma$ 
is a 0-simplex, then there is only one class and hence only one orientation of $\sigma$.) 
An oriented simplex is a simplex $\sigma$ together with an orientation of $\sigma$.

If the points $v_{0}, \ldots, v_{p}$ are independent, we shall use the symbol

$$v_{0} \ldots v_{p}$$
to denote the simplex they span, and we shall use the symbol

$$[v_{0}, \ldots, v_{p}]$$
to denote the oriented simplex consisting of the simplex $v_{0} \ldots v_{p}$ and the equiva-
ence class of the particular ordering $(v_{0}, \ldots, v_{p})$. 
§11. The Computability of Homology Groups

We have computed the homology groups of some familiar spaces, such as the sphere and the torus and the Klein bottle. Now we ask the question whether one can in fact compute homology groups in general. For finite complexes, the answer is affirmative. In this section, we present an explicit algorithm for carrying out the computation.

First, we prove a basic theorem giving a "normal form" for homomorphisms of finitely generated free abelian groups. The proof is constructive in nature. One corollary is the theorem about subgroups of free abelian groups that we stated earlier as Theorem 4.2. A second corollary is a theorem concerning standard bases for free chain complexes. And a third corollary gives our desired algorithm for computing the homology groups of a finite complex.

First, we need two lemmas with which you might already be familiar.

Lemma 11.1. Let $A$ be a free abelian group of rank $n$. If $B$ is a subgroup of $A$, then $B$ is free abelian of rank $r \leq n$.

\textbf{Proof.} We may without loss of generality assume that $B$ is a subgroup of the $n$-fold direct product $Z^n = Z \times \cdots \times Z$. We construct a basis for $B$ as follows:

Let $\pi_i : Z^n \rightarrow Z$ denote projection on the $i$th coordinate. For each $m \leq n$, let $B_m$ be the subgroup of $B$ defined by the equation

$$B_m = B \cap (Z^m \times 0).$$

That is, $B_m$ consists of all $x \in B$ such that $\pi_i(x) = 0$ for $i > m$. In particular, $B_n = B$. Now the homomorphism

$$\pi_m : B_m \rightarrow Z$$

carries $B_m$ onto a subgroup of $Z$. If this subgroup is trivial, let $x_m = 0$; otherwise, choose $x_m \in B_m$ so that its image $\pi_m(x_m)$ generates this subgroup. We assert that the non-zero elements of the set $\{x_1, \ldots, x_n\}$ form a basis for $B$.

First, we show that for each $m$, the elements $x_1, \ldots, x_m$ generate $B_m$. (Then, in particular, the elements $x_1, \ldots, x_n$ generate $B$.) It is trivial that $x_1$ generates $B_1$; indeed if $d$ is the integer $\pi_1(x_1)$, then

$$x_1 = (d, 0, \ldots, 0)$$

and $B_1$ consists of all multiples of this element.

Assume that $x_1, \ldots, x_{m-1}$ generate $B_{m-1}$; let $x \in B_m$. Now $\pi_m(x) = k\pi_m(x_m)$ for some integer $k$. It follows that

$$\pi_m(x - kx_m) = 0,$$

so that $x - kx_m$ belongs to $B_{m-1}$. Then

$$x - kx_m = k_1x_1 + \cdots + k_{m-1}x_{m-1}$$

by the induction hypothesis. Hence $x_1, \ldots, x_m$ generate $B_m$. 

Second, we show that for each \( m \), the non-zero elements in the set \( \{x_1, \ldots, x_m\} \) are independent. The result is trivial when \( m = 1 \). Suppose it true for \( m - 1 \). Then we show that if

\[
\lambda_1 x_1 + \cdots + \lambda_m x_m = 0,
\]

then it follows that for each \( i \), \( \lambda_i = 0 \) whenever \( x_i \neq 0 \); independence follows.

Applying the map \( \pi_m \), we derive the equation

\[
\lambda_m \pi_m(x_m) = 0.
\]

From this equation, it follows that either \( \lambda_m = 0 \) or \( x_m = 0 \). For if \( \lambda_m \neq 0 \), then \( \pi_m(x_m) = 0 \), whence the subgroup \( \pi_m(B_m) \) is trivial and \( x_m = 0 \) by definition. We conclude two things:

\[
\lambda_m = 0 \quad \text{if} \quad x_m \neq 0,
\]

\[
\lambda_1 x_1 + \cdots + \lambda_{m-1} x_{m-1} = 0.
\]

The induction hypothesis now applies to show that for \( i < m \),

\[
\lambda_i = 0 \quad \text{whenever} \quad x_i \neq 0. \quad \square
\]

For later use, we generalize this result to arbitrary free abelian groups:

**Lemma 11.2.** If \( A \) is a free abelian group, any subgroup \( B \) of \( A \) is free.

**Proof.** The proof given for the finite case generalizes, provided we assume that the basis for \( A \) is indexed by a well-ordered set \( J \) having a largest element. (And the well-ordering theorem, which is equivalent to the axiom of choice, tells us this assumption is justified.)

We begin by assuming \( A \) equals a direct sum of copies of \( \mathbb{Z} \); that is, \( A \) equals the subgroup of the cartesian product \( \mathbb{Z}' \) consisting of all tuples \((n_\alpha)_{\alpha \in J}\) such that \( n_\alpha = 0 \) for all but finitely many \( \alpha \). Then we proceed as before.

Let \( B \) be a subgroup of \( A \). Let \( B_\beta \) consist of those elements \( x \) of \( B \) such that \( \pi_\alpha(x) = 0 \) for \( \alpha > \beta \). Consider the subgroup \( \pi_\beta(B_\beta) \) of \( \mathbb{Z} \); if it is trivial define \( x_\beta = 0 \), otherwise choose \( x_\beta \in B_\beta \) so \( \pi_\beta(x_\beta) \) generates the subgroup.

We show first that the set \( \{x_\alpha \mid \alpha \leq \beta\} \) generates \( B_\beta \). This fact is trivial if \( \beta \) is the smallest element of \( J \). We prove it in general by transfinite induction. Given \( x \in B_\beta \), we have

\[
\pi_\beta(x) = k \pi_\beta(x_\beta)
\]

for some integer \( k \). Hence \( \pi_\beta(x - kx_\beta) = 0 \). Consider the set of those indices \( \alpha \) for which \( \pi_\alpha(x - kx_\beta) \neq 0 \). (If there are none, \( x = kx_\beta \) and we are through.) All of these indices are less than \( \beta \), because \( x \) and \( x_\beta \) belong to \( B_\beta \). Furthermore, this set of indices is finite, so it has a largest element \( \gamma \), which is less than \( \beta \). But this means that \( x - kx_\beta \) belongs to \( B_\gamma \), whence by the induction hypothesis, \( x - kx_\beta \) can be written as a linear combination of elements \( x_\alpha \) with each \( \alpha \leq \gamma \).

Second, we show that the non-zero elements in the set \( \{x_\alpha \mid \alpha \leq \beta\} \) are inde-
pendent. Again, this fact is trivial if $\beta$ is the smallest element of $J$. In general, suppose
\[\lambda_{\alpha_1}x_{\alpha_1} + \cdots + \lambda_{\alpha_k}x_{\alpha_k} + \lambda_{\beta}x_{\beta} = 0,\]
where $\alpha_i < \beta$. Applying $\pi_{\beta}$, we see that
\[\lambda_{\beta} \pi_{\beta}(x_{\beta}) = 0.\]
As before, it follows that either $\lambda_{\beta} = 0$ or $x_{\beta} = 0$. We conclude that
\[\lambda_{\beta} = 0 \quad \text{if} \quad x_{\beta} \neq 0,\]
and
\[\lambda_{\alpha_i}x_{\alpha_i} + \cdots + \lambda_{\alpha_k}x_{\alpha_k} = 0.\]
The induction hypothesis now implies that $\lambda_{\alpha_i} = 0$ whenever $x_{\alpha_i} \neq 0$. \(\square\)

We now prove our basic theorem. First we need a definition.

**Definition.** Let $G$ and $G'$ be free abelian groups with bases $a_1, \ldots, a_n$ and $a'_1, \ldots, a'_m$, respectively. If $f : G \to G'$ is a homomorphism, then
\[f(a_j) = \sum_{i=1}^{m} \lambda_{ij}a'_i\]
for unique integers $\lambda_{ij}$. The matrix $(\lambda_{ij})$ is called the **matrix of $f$** relative to the given bases for $G$ and $G'$.

**Theorem 11.3.** Let $G$ and $G'$ be free abelian groups of ranks $n$ and $m$, respectively; let $f : G \to G'$ be a homomorphism. Then there are bases for $G$ and $G'$ such that, relative to these bases, the matrix of $f$ has the form
\[
B = \begin{bmatrix}
  b_1 & 0 & \cdots & 0 \\
  \ddots & \ddots & \ddots & \ddots \\
  0 & b_i & \cdots & b_i \\
  0 & 0 & \cdots & 0
\end{bmatrix}
\]
where $b_i \geq 1$ and $b_1 \mid b_2 \mid \cdots \mid b_i$.

This matrix is in fact uniquely determined by $f$ (although the bases involved are not). It is called a **normal form** for the matrix of $f$.

**Proof.** We begin by choosing bases in $G$ and $G'$ arbitrarily. Let $A$ be the matrix of $f$ relative to these bases. We shall give shortly a procedure for modify-
ing these bases so as to bring the matrix into the normal form described. It is called “the reduction algorithm.” The theorem follows. □

Consider the following “elementary row operations” on an integer matrix $A$:

1. Exchange row $i$ and row $k$.
2. Multiply row $i$ by $-1$.
3. Replace row $i$ by $(\text{row } i) + q(\text{row } k)$, where $q$ is an integer and $k \neq i$.

Each of these operations corresponds to a change of basis in $G'$. The first corresponds to an exchange of $a'_i$ and $a'_k$. The second corresponds to replacing $a'_i$ by $-a'_i$. And the third corresponds to replacing $a'_k$ by $a'_k - qa'_i$, as you can readily check.

There are three similar “column operations” on $A$ that correspond to changes of basis in $G$.

We now show how to apply these six operations to an arbitrary matrix $A$ so as to reduce it to our desired normal form. We assume $A$ is not the zero matrix, since in that case the result is trivial.

Before we begin, we note the following fact: If $c$ is an integer that divides each entry of the matrix $A$, and if $B$ is obtained from $A$ by applying any one of these elementary operations, then $c$ also divides each entry of $B$.

The reduction algorithm

Given a matrix $A = (a_{ij})$ of integers, not all zero, let $\alpha(A)$ denote the smallest non-zero element of the set of numbers $|a_{ij}|$. We call $a_{ij}$ a minimal entry of $A$ if $|a_{ij}| = \alpha(A)$.

The reduction procedure consists of two steps. The first brings the matrix to a form where $\alpha(A)$ is as small as possible. The second reduces the dimensions of the matrix involved.

**Step 1.** We seek to modify the matrix by elementary operations so as to decrease the value of the function $\alpha$. We prove the following:

*If the number $\alpha(A)$ fails to divide some entry of $A$, then it is possible to decrease the value of $\alpha$ by applying elementary operations to $A$; and conversely.*

The converse is easy. If the number $\alpha(A)$ divides each entry of $A$, then it will divide each entry of any matrix $B$ obtained by applying elementary operations to $A$. In this situation, it is not possible to reduce the value of $\alpha$ by applying elementary operations.

To prove the result itself, we suppose $a_{ij}$ is a minimal entry of $A$ that fails to divide some entry of $A$. If the entry $a_{ij}$ fails to divide some entry $a_{kj}$ in its column, then we perform a division, writing

$$\frac{a_{kj}}{a_{ij}} = q + \frac{r}{a_{ij}},$$
where \(0 < |r| < |a_{ij}|\). Signs do not matter here; \(q\) and \(r\) may be either positive or negative. We then replace (row \(k\)) of \(A\) by (row \(k\)) \(- q\) (row \(i\)). The result is to replace the entry \(a_{kj}\) in the \(k\)th row and \(j\)th column of \(A\) by \(a_{kj} - qa_{ij} = r\). The value of \(\alpha\) for this new matrix is at most \(|r|\), which is less than \(\alpha(A)\).

A similar argument applies if \(a_{ij}\) fails to divide some entry in its row.

Finally, suppose \(a_{ij}\) divides each entry in its row and each entry in its column, but fails to divide the entry \(a_{st}\), where \(s \neq i\) and \(t \neq j\). Consider the following four entries of \(A\):

\[
\begin{align*}
a_{ij} & \quad \cdots \quad a_{it} \\
\quad \vdots & \quad \quad \vdots \\
a_{sj} & \quad \cdots \quad a_{st}
\end{align*}
\]

Because \(a_{ij}\) divides \(a_{ij}\), we can by elementary operations bring the matrix to the form where the entries in these four places are as follows:

\[
\begin{align*}
a_{ij} & \quad \cdots \quad a_{it} \\
\quad \vdots & \quad \quad \vdots \\
0 & \quad \cdots \quad a_{st} + la_{it}
\end{align*}
\]

If we then replace (row \(i\)) of this matrix by (row \(i\)) + (row \(s\)), we are back in the previous situation, where \(a_{ij}\) fails to divide some entry in its row.

**Step 2.** At the beginning of this step, we have a matrix \(A\) whose minimal entry divides every entry of \(A\).

Apply elementary operations to bring a minimal entry of \(A\) to the upper left corner of the matrix and to make it positive. Because it divides all entries in its row and column, we can apply elementary operations to make all the other entries in its row and column into zeros. Note that at the end of this process, the entry in the upper left corner divides all entries of the matrix.

One now begins Step 1 again, applying it to the smaller matrix obtained by ignoring the first row and first column of our matrix.

**Step 3.** The algorithm terminates either when the smaller matrix is the zero matrix or when it disappears. At this point our matrix is in normal form. The only question is whether the diagonal entries \(b_1, \ldots, b_l\) successively divide one another. But this is immediate. We just noted that at the end of the first application of Step 2, the entry \(b_i\) in the upper left corner divides all entries of the matrix. This fact remains true as we continue to apply elementary operations. In particular, when the algorithm terminates, \(b_i\) must divide each of \(b_{i+1}, \ldots, b_l\).

A similar argument shows \(b_2\) divides each of \(b_3, \ldots, b_l\). And so on.

It now follows immediately from Exercise 4 of §4 that the numbers \(b_1, \ldots, b_l\) are uniquely determined by the homomorphism \(f\). For the number \(l\) of non-zero entries in the matrix is just the rank of the free abelian group \(f(G) \subseteq G'\). And those numbers \(b_i\) that are greater than 1 are just the torsion coefficients \(t_1, \ldots, t_k\) of the quotient group \(G'/f(G)\).
Applications of the reduction algorithm

Now we prove the basic theorem concerning subgroups of free abelian groups, which we stated in §4.

Proof of Theorem 4.2. Given a free abelian group $F$ of rank $n$, we know from Lemma 11.1 that any subgroup $R$ is free of rank $r \leq n$. Consider the inclusion homomorphism $j : R \to F$, and choose bases $a_1, \ldots, a_r$ for $R$ and $e_1, \ldots, e_n$ for $F$ relative to which the matrix of $j$ is in the normal form of the preceding theorem. Because $j$ is a monomorphism, this normal form has no zero columns. Thus $j(a_i) = b_i e_i$ for $i = 1, \ldots, r$, where $b_i \geq 1$ and $b_1 \mid b_2 \mid \cdots \mid b_r$. Since $j(a_i) = a_i$, it follows that $b_1 e_1, \ldots, b_r e_r$ is a basis for $R$. $\square$

Now we prove the "standard basis theorem" for free chain complexes.

Definition. A chain complex $\mathcal{C}$ is a sequence

$$\ldots \to C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \to \ldots$$

of abelian groups $C_i$ and homomorphisms $\partial_i$, indexed with the integers, such that $\partial_p \circ \partial_{p+1} = 0$ for all $p$. The $p$th homology group of $\mathcal{C}$ is defined by the equation

$$H_p(\mathcal{C}) = \ker \partial_p / \text{im} \partial_{p+1}.$$ 

If $H_p(\mathcal{C})$ is finitely generated, its betti number and torsion coefficients are called the betti number and torsion coefficients of $\mathcal{C}$ in dimension $p$.

Theorem 11.4 (Standard bases for free chain complexes). Let $\{C_p, \partial_p\}$ be a chain complex; suppose each group $C_p$ is free of finite rank. Then for each $p$ there are subgroups $U_p, V_p, W_p$ of $C_p$ such that

$$C_p = U_p \oplus V_p \oplus W_p,$$

where $\partial_p(U_p) \subset W_{p-1}$ and $\partial_p(V_p) = 0$ and $\partial_p(W_p) = 0$. Furthermore, there are bases for $U_p$ and $W_{p-1}$ relative to which $\partial_p : U_p \to W_{p-1}$ has a matrix of the form

$$B = \begin{bmatrix} b_1 & & 0 \\ & \ddots & \vdots \\ 0 & & b_l \end{bmatrix},$$

where $b_i \geq 1$ and $b_1 \mid b_2 \mid \cdots \mid b_l$.

Proof. Step 1. Let

$$Z_p = \ker \partial_p \quad \text{and} \quad B_p = \text{im} \partial_{p+1}.$$ 

Let $W_p$ consist of all elements $c_p$ of $C_p$ such that some non-zero multiple of $c_p$
belongs to $B_p$. It is a subgroup of $C_p$, and is called the group of **weak boundaries.** Clearly

$$B_p \subset W_p \subset Z_p \subset C_p.$$  

(The second inclusion uses the fact that $C_p$ is torsion-free, so that the equation $mc_p = \partial_{p+1}d_{p+1}$ implies that $\partial_p c_p = 0$.) We show that $W_p$ is a direct summand in $Z_p$.

Consider the natural projection

$$Z_p \to H_p(\mathcal{O}) \to H_p(\mathcal{O})/T_p(\mathcal{O}),$$

where $T_p(\mathcal{O})$ is the torsion subgroup of $H_p(\mathcal{O})$. The kernel of this projection is $W_p$; therefore, $Z_p/W_p \cong H_p/T_p$. The latter group is finitely generated and torsion-free, so it is free. If $c_1 + W_p, \ldots, c_k + W_p$ is a basis for $Z_p/W_p$, and $d_1, \ldots, d_l$ is a basis for $W_p$, then it is straightforward to check that $c_1, \ldots, c_k, d_1, \ldots, d_l$ is a basis for $Z_p$. Then $Z_p = V_p \oplus W_p$, where $V_p$ is the group with basis $c_1, \ldots, c_k$.

**Step 2.** Suppose we choose bases $e_1, \ldots, e_n$ for $C_p$, and $e'_1, \ldots, e'_m$ for $C_{p-1}$, relative to which the matrix of $\partial_p : C_p \to C_{p-1}$ has the normal form

$$
\begin{pmatrix}
  e_1 & \cdots & e_l & e_{l+1} & \cdots & e_n \\
  e'_1 \\
  \vdots \\
  e'_l \\
  e'_{l+1} \\
  \vdots \\
  e'_m
\end{pmatrix}
\begin{pmatrix}
  b_1 & 0 & 0 \\
  \vdots & \ddots & \vdots \\
  0 & b_l & 0
\end{pmatrix}
$$

where $b_i \geq 1$ and $b_1 \mid b_2 \mid \cdots \mid b_l$. Then the following hold:

1. $e_{l+1}, \ldots, e_n$ is a basis for $Z_p$.
2. $e'_1, \ldots, e'_l$ is a basis for $W_{p-1}$.
3. $b_1 e'_1, \ldots, b_l e'_l$ is a basis for $B_{p-1}$.

We prove these results as follows: Let $c_p$ be the general $p$-chain. We compute its boundary; if

$$c_p = \sum_{i=1}^n a_i e_i,$$

then

$$\partial_p c_p = \sum_{i=1}^l a_i b_i e'_i.$$  

To prove (1), we note that since $b_i \neq 0$, the $p$-chain $c_p$ is a cycle if and only if $a_i = 0$ for $i = 1, \ldots, l$. To prove (3), we note that any $p-1$ boundary $\partial_p c_p$ lies in the group generated by $b_1 e'_1, \ldots, b_l e'_l$, since $b_i \neq 0$, these elements are inde-
pendent. Finally, we prove (2). Note first that each of $e'_1, \ldots, e'_i$ belongs to $W_{p-1}$, since $b_i e'_i = \partial e_i$. Conversely, let
\[ c_{p-1} = \sum_{i=1}^{m} d_i e'_i \]
be a $p-1$ chain and suppose $c_{p-1} \in W_{p-1}$. Then $c_{p-1}$ satisfies an equation of the form
\[ \lambda c_{p-1} = \delta_p c_p = \sum_{i=1}^{l} a_i b_i e'_i \]
for some $\lambda \neq 0$. Equating coefficients, we see that $\lambda d_i = 0$ for $i > l$, whence $d_i = 0$ for $i > l$. Thus $e'_1, \ldots, e'_i$ is a basis for $W_{p-1}$.

**Step 3.** We prove the theorem. Choose bases for $C_p$ and $C_{p-1}$ as in Step 2. Define $U_p$ to be the group spanned by $e_1, \ldots, e_i$; then
\[ C_p = U_p \oplus Z_p. \]
Using Step 1, choose $V_p$ so that $Z_p = V_p \oplus W_p$. Then we have a decomposition of $C_p$ such that $\partial_p (V_p) = 0$ and $\partial_p (W_p) = 0$. The existence of the desired bases for $U_p$ and $W_{p-1}$ follows from Step 2. \( \square \)

Note that $W_p$ and $Z_p = V_p \oplus W_p$ are uniquely determined subgroups of $C_p$. The subgroups $U_p$ and $V_p$ are not uniquely determined, however.

**Theorem 11.5.** The homology groups of a finite complex $K$ are effectively computable.

**Proof.** By the preceding theorem, there is a decomposition
\[ C_p(K) = U_p \oplus V_p \oplus W_p \]
where $Z_p = V_p \oplus W_p$ is the group of $p$-cycles and $W_p$ is the group of weak $p$-boundaries. Now
\[ H_p(K) = Z_p/B_p \cong V_p \oplus (W_p/B_p) \cong (Z_p/W_p) \oplus (W_p/B_p). \]
The group $Z_p/W_p$ is free and the group $W_p/B_p$ is a torsion group; computing $H_p(K)$ thus reduces to computing these two groups.

Let us choose bases for the chain groups $C_p(K)$ by orienting the simplices of $K$, once and for all. Then consider the matrix of the boundary homomorphism $\partial_p : C_p(K) \to C_{p-1}(K)$ relative to this choice of bases; the entries of this matrix will in fact have values in the set $\{0, 1, -1\}$. Using the reduction algorithm described earlier, we reduce this matrix to normal form. Examining Step 2 of the preceding proof, we conclude from the results proved there the following facts about this normal form:
(1) The rank of $Z_p$ equals the number of zero columns.

(2) The rank of $W_{p-1}$ equals the number of non-zero rows.

(3) There is an isomorphism

$$W_{p-1}/B_{p-1} \cong \mathbb{Z}/b_1 \oplus \mathbb{Z}/b_2 \oplus \cdots \oplus \mathbb{Z}/b_l.$$ 

Thus the normal form for the matrix of $\partial_p: C_p \to C_{p-1}$ gives us the torsion coefficients of $K$ in dimension $p-1$; they are the entries of the matrix that are greater than 1. This normal form also gives us the rank of $Z_p$. On the other hand, the normal form for $\partial_{p+1}: C_{p+1} \to C_p$ gives us the rank of $W_p$. The difference of these numbers is the rank of $Z_p/W_p$—that is, the betti number of $K$ in dimension $p$. □

**EXERCISES**

1. Show that the reduction algorithm is not needed if one wishes merely to compute the betti numbers of a finite complex $K$; instead all that is needed is an algorithm for determining the rank of a matrix. Specifically, show that if $A_p$ is the matrix of $\partial_p : C_p(K) \to C_{p-1}(K)$ relative to some choice of basis, then

$$\beta_p(K) = \text{rank } C_p(K) - \text{rank } A_p - \text{rank } A_{p+1}.$$  

2. Compute the homology groups of the quotient space indicated in Figure 11.1. [Hint: First check whether all the vertices are identified.]

3. Reduce to normal form the matrix

$$\begin{bmatrix} 2 & 6 & 4 \\ 4 & -7 & 4 \\ 4 & 8 & 4 \end{bmatrix}.$$