1. Problem 2

For \( t \in [0, 1] \), let \( f_t : S^1 \times I \to S^1 \times I \) be defined by \( f_t(\theta, s) = (\theta + 2\pi st, s) \). Then \( f_0 = id \) and \( f_1 = f \). Moreover, \( f_t \) restricted to \( S^1 \times \{0\} \) is the identity map.

Glue \( S^1 \times \{0\} \) to \( S^1 \times \{1\} \) by \( (\theta, 0) \sim (\theta, 1) \) to get a torus \( T^2 \). Note that the map \( f \) descends down to a map \( F \) of the torus. If there is a homotopy \( f_t \) between \( f \) and \( id \) that is identity restricted to \( S^1 \times \{0\} \) and \( S^1 \times \{1\} \), then it descends down to a homotopy \( F_t \) of \( F \) to the identity map on the torus. This means that the map \( F_* \) induced on \( \pi_1(T^2) \) has to be identity.

The path \( \gamma_0 : s \to (\theta_0, s) \) maps down to the longitude of the torus. The image \( F_*(\gamma_0) \neq \gamma_0 \) (in fact, it is the \((1, 1)\) curve on the torus). Thus a homotopy as above cannot exist.

2. Problem 3

Let \( \alpha \) be a loop based at \( x_0 \) in \( X \) and let \( \beta \) be a loop based at \( y_0 \) in \( Y \). Let \( \delta \) be the loop in \( X \times Y \) given by the concatenation of the loops \( \alpha \times \{y_0\} \) and \( \{x_0\} \times \beta \) in that order i.e. first \( \alpha \times \{y_0\} \) then \( \{x_0\} \times \beta \). Let \( \eta \) be the loop in \( X \times Y \) given by concatenating in the reverse order.

Consider the 1-parameter family of loops \( \gamma_t : [-1, 2] \to X \times Y \) given by

\[
\gamma_t(s) = \begin{cases} 
(\alpha(s), \beta(t)) & s \in [0, 1] \\
(x_0, \beta((1-t)(s-2) + 1)) & s \in [1, 2]
\end{cases}
\]

In words, the loop \( \gamma_t \) follows \( \beta \) from \((x_0, y_0)\) to \((x_0, \beta(t))\), then does the loop \((\alpha, \beta(t))\) in \( X \times \{\beta(t)\} \) and then follows the rest of \( \beta \) from \((x_0, \beta(t))\) back to \((x_0, y_0)\). It is easy to check that this gives a homotopy between \( \delta \) and \( \eta \) i.e. \( \gamma_0 = \delta \) and \( \gamma_1 = \eta \).

3. Problem 5

Draw the torus as the square in \( \mathbb{R}^2 \) with opposite sides identified and with vertices \((\pm 1, \pm 1)\). The union of the longitude and meridian circles of the torus can be taken to be the boundary of this square. Take the origin in \( \mathbb{R}^2 \) to be the deleted point of the torus. Let \(|(x, y)| = \max\{x, y\} \), and for \( t \in [0, 1] \) let \( s = 1 - t(1 - |(x, y)|) \). Consider the deformation retract

\[
F_t(x, y) = \frac{1}{s}(x, y)
\]

Check that \( F_0 \) is the identity on the punctured torus and \( F_1(x, y) \) belongs to the boundary of the square.