The Mayer-Vietoris sequence gives

\[ \cdots \to H_n(M_g) \to H_n(R) \oplus H_n(R) \to H_n(X) \to H_{n-1}(M_g) \to \cdots \]

The compact space \( R \) deformation retracts on a wedge of \( g \) circles. So \( H_0(R) = \mathbb{Z} \), \( H_1(R) = \bigoplus_{i=1}^{g} \mathbb{Z} \), \( H_n(R) = 0 \) for \( n > 1 \). Also we know \( H_0(M_g) = \mathbb{Z} \), \( H_1(M_g) = \bigoplus_{i=1}^{2g} \mathbb{Z} \), \( H_2(M_g) = \mathbb{Z} \), \( H_n(R) = 0 \) for \( n > 1 \). Call the 1-cycles on \( M_g \) that give the generators of \( H_1(R) \) as the \( a \)-cycles. Then, there are the 1-cycles on \( M_g \) that bound disks in \( R \), which we shall call the \( b \)-cycles. The \( a \\ and \( b \)-cycles together generate \( H_1(M_g) \) and the induced maps \( H_1(M_g) \to H_1(R) \) send the \( a \)-cycles to the generators of \( H_1(R) \) for both the handle-bodies and send the \( b \)-cycles to zero. By computing the various maps in the Mayer-Vietoris sequence, in particular, the map \( H_1(M_g) \to H_1(R) \oplus H_1(R) \) has the diagonal as the image, with kernel \( \bigoplus_{i=1}^{g} \mathbb{Z} \) generated by the \( b \)-cycles, we get

1. Since \( H_n(R) = H_n(M_g) = 0 \) for all \( n \geq 3 \), we get \( H_n(X) = 0 \) for all \( n > 3 \).
2. Because \( H_3(R) = H_2(R) = 0 \), we get \( H_3(X) = H_2(M_g) = \mathbb{Z} \).
3. Because \( H_2(R) = 0 \), we have the exact sequence

\[ 0 \to H_2(X) \to H_1(M_g) \to H_1(R) \oplus H_1(R) \to \cdots \]

which implies \( H_2(X) = Ker(H_1(M_g) \to H_1(R) \oplus H_1(R)) = \bigoplus_{i=1}^{g} \mathbb{Z} \) generated by the \( b \)-cycles.
4. The Mayer-Vietoris sequence terminates in

\[ \cdots \to H_1(X) \to H_0(M_g) \to H_0(R) \oplus H_0(R) \to H_0(X) \to 0 \]

This implies the map \( H_0(R) \oplus H_0(R) \to H_0(X) \) is surjective, and as before the map \( H_0(M_g) \to H_0(R) \oplus H_0(R) \) has image the diagonal. Hence \( H_0(X) = H_0(R) \oplus H_0(R)/\Delta = \mathbb{Z} \).
5. The map \( H_0(M_g) \to H_0(R) \oplus H_0(R) \) is also injective, so the image of \( H_1(X) \to H_0(M_g) \) is just 0. This gives us the exact sequence

\[ 0 \to H_2(X) \to H_1(M_g) \to H_1(R) \oplus H_1(R) \to H_1(X) \to 0 \]

In the above sequence, the image of \( H_1(M_g) \) is the diagonal. So \( H_1(X) = H_1(R) \oplus H_1(R)/\Delta = \bigoplus_{i=1}^{g} \mathbb{Z} \).
To find the relative homology groups $H_n(R, M_g)$, we use the long exact sequence in homology

$$
\cdots \rightarrow H_n(M_g) \rightarrow H_n(R) \rightarrow H_n(R, M_g) \rightarrow H_{n-1}(M_g) \rightarrow \cdots
$$

This gives

1. $H_n(R, M_g) = 0$ for $n > 3$ because $H_n(R) = H_n(M_g) = 0$ for $n \geq 3$.
2. $H_2(R) = 0$ implies $H_3(R, M_g) = H_2(M_g) = \mathbb{Z}$ and we get the exact sequence

$$
0 \rightarrow H_2(R, M_g) \rightarrow H_1(M_g) \rightarrow H_1(R) \rightarrow \cdots
$$

which means that $H_2(R, M_g) \rightarrow H_1(M_g)$ is injective.
3. The long exact sequence terminates in

$$
\cdots \rightarrow H_1(R, M_g) \rightarrow H_0(M_g) \rightarrow H_0(R) \rightarrow H_0(R, M_g) \rightarrow 0
$$

The map $H_0(M_g) \rightarrow H_0(R)$ is an isomorphism, so $H_0(R, M_g) = 0$. Also, the isomorphism gives the exact sequence

$$
0 \rightarrow H_2(R, M_g) \rightarrow H_1(M_g) \rightarrow H_1(R) \rightarrow H_1(R, M_g) \rightarrow 0
$$

The map $H_1(M_g) \rightarrow H_1(R)$ is onto, so $H_1(R, M_g) = 0$. So $H_2(R, M_g) = Ker(H_1(M_g) \rightarrow H_1(R)) = \oplus_{i=1}^{g} \mathbb{Z}$.

**Remark 0.1.** Instead of gluing by the two handle-bodies by the identity map of the boundary surface, if we glue by a map of the surface that interchanges the $a$ and $b$-cycles, we get $S^3$ (embed the genus $g$ handle-body in $\mathbb{R}^3$ in the standard way; check that the complement of it in $S^3$ is a genus $g$ handle-body with the $a$ and $b$-cycles switched). In fact, any closed 3-manifold can be obtained by gluing two handle-bodies by a map of the boundary surface. This decomposition is called a Heegard decomposition of the 3-manifold.

**Hatcher 2.2 Problem 35**

Suppose there is a an embedding of $X$ in $\mathbb{R}^3$ with a neighborhood $U$, a mapping cylinder of a map $f : S \rightarrow X$, where $S$ is a closed orientable surface. Let $V = S^3 \setminus X$ and write $S^3 = U \cup V$. By the Mayer-Vietoris sequence

$$
\cdots \rightarrow H_2(S^3) \rightarrow H_1(U \cap V) \rightarrow H_1(U) \oplus H_1(V) \rightarrow H_1(S^3) \rightarrow \cdots
$$

Since $H_2(S^3) = H_1(S^3) = 1$, this says that there is an isomorphism $H_1(U \cap V) = H_1(U) \oplus H_1(V)$. Note that since $U$ is the mapping cylinder of $f$, it deformation retracts to $X$. The set $U \cap V = S \times [0,1)$ and so deformation retracts to $S$. So if $H_1(X)$ has torsion we get a contradiction.

**Problem 4**

**(a):** The set of 2-chains, $C_2(\Sigma)$ is generated by the triangles $T_i$, and the 1-chains, $C_1(\Sigma)$ by the edges $e_i$. Suppose a non-zero element $\sum a_i T_i$ is in the kernel of $\partial : C_2(\Sigma) \rightarrow C_1(\Sigma)$. Since each edge is shared by exactly two adjacent triangles, the coefficient $a_1$ determines completely all other coefficients $a_i$ (in fact, all other coefficients $a_i = \pm a_1$). So the kernel is generated by a single element. Since $C_2(\Sigma) = 0$, the second homology $H_2(\Sigma)$ is isomorphic to the kernel in $C_2(\Sigma)$, so in this case it is $\mathbb{Z}$. The other possibility is that there is no non-zero element in the kernel, in which case $H_2(\Sigma) = 0$. 

2
(b): There is obviously a map of any degree from $T^2 \rightarrow T^2$. So it is enough to show that there is a map of degree 1 from $T^2 \rightarrow S^2$. The torus $T^2$ is a square with opposite sides identified. Then, think of $S^2$ as the square with the entire boundary identified to a single point. This gives the degree 1 map from $T^2 \rightarrow S^2$.

On the other hand, let $f : S^2 \rightarrow T^2$. Since $S^2$ is simply connected, $f$ lifts to the universal cover i.e. to a map $F : S^2 \rightarrow \mathbb{R}^2$. Since $\mathbb{R}^2$ is contractible, there is a homotopy of $f$ to the constant map. This implies that the degree of $f$ is zero.