1. Problem 1

Let $C$ be a closed subset of $Y$. Then $f^{-1}(C) = f_A^{-1}(C) \cup f_B^{-1}(C)$. Since $f_A$ and $f_B$ are continuous, the sets $f_A^{-1}(C)$ and $f_B^{-1}(C)$ are closed in $A$ and $B$ respectively, and hence in $X$. This implies $f^{-1}(C)$ is closed in $X$, and hence $f$ is continuous.

2. Problem 4

(a) ⇒ (b). Let $f : S^1 \to X$ be a map and suppose $f$ is homotopic to the constant map $c_0 : S^1 \to x_0$. This means that there is a map $F : S^1 \times [0, 1] \to X$ such that $F(a, 0) = f(a)$ and $F(a, 1) = x_0$. Consider the quotient map $q : S^1 \times [0, 1] \to \mathbb{D}^2$ given by identifying all points $(a, 1)$ to a single point. Because $F(a, 1) = x_0$, the map descends to a map $\overline{F} : \mathbb{D}^2 \to X$ such that $\overline{F} = F \circ q$. The map $\overline{F}$ restricted to $S^1 = q(S^1 \times \{0\})$ is $f$.

(b) ⇒ (c). Embed $S^1 \subseteq \mathbb{C}$ in the standard way. Think of a loop based at $x_0$ as a map $f : S^1 \to X$ that sends $1 \in S^1$ to $x_0$. Extend it to a map $F : \mathbb{D}^2 \to X$. Consider the map $G : S^1 \times [0, 1] \to X$ defined by $G = F \circ q$. The map $G$ defines a homotopy to the constant map $F(0)$. Since any two constant maps are homotopic and homotopy is a transitive relation, $\pi_1(X, x_0) = 0$.

(c) ⇒ (a). Directly from definitions.

When $X$ is simply connected, every map $f$ from $S^1$ into $X$ thought of as a loop in $X$ based at $x_0 = f(1)$ is homotopic to the constant map to $x_0$. Similarly, $g$ from $S^1$ into $X$ is homotopic to the constant map to $g(1)$. Any two constant maps are homotopic. Since homotopy is a transitive relation, this implies $f$ and $g$ are homotopic. The converse is just the definition.