HW: From printout.

Next time: Move on 7.3, 7.4.

Scheey: Div, grad, curl are all that.

Last time: Parameterizing surfaces

\[ r: (D \subset \mathbb{R}^2) \longrightarrow (S \subset \mathbb{R}^3) \]

Q: What is the area of a surface?
We can approximate the area of region at the parallelogram shown right by

\[ \Delta v T_v \]
\[ \Delta u T_u \]

as you can see from this diagram.

\[ r(u_0, v_0) \]

\[ r(u_0 + \Delta u, v_0) = r(u_0, v_0) + \Delta u \frac{\partial r}{\partial u} |_{(u_0,v_0)} \]

\[ r(u_0 + \Delta u, v_0) \approx \Delta u T_u \]

The area of this parallelogram is

\[ \| (\Delta u T_u) \times (\Delta v T_v) \| = \| T_u \times T_v \| \Delta u \Delta v. \]

Thus the area of \( S \) is

\[ \sum_{\text{small squares}} \| T_u \times T_v \| \Delta u \Delta v \]

which implies

\[ \text{Area}(S) = \iint_D \| T_u \times T_v \| \, du \, dv \]
We can think of $\|T_\phi \times T_\theta\|$ as a change of area factor, just as in 2D change of variables. [Cf. Distortion of area in map projections.]

**Ex:** Unit sphere

$$D = \begin{bmatrix} \frac{\partial R}{\partial \phi} \times \frac{\partial R}{\partial \theta} \end{bmatrix}$$

$$R(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$$

$$T_\phi \times T_\theta = \det \begin{pmatrix} i & j & k \\ -T_\phi & -T_\theta \end{pmatrix}$$

$$= (\sin^2 \phi \cos \theta \sin \phi, \sin \phi \sin \theta, \sin \phi \cos \phi)$$

$$\|T_\phi \times T_\theta\| = \sqrt{\sin^4 \phi + \sin^2 \phi \cos^2 \phi} = \sin \phi$$

Geometrically, we saw this when we were messing with spherical coordinates.
Since

has area \( \approx \sin \phi \Delta \phi \Delta \theta \).

Either way, we have

\[
\text{Area} (\bigcirc) = \iiint_D \sin \phi \, d\phi \, d\theta \]

\[
= \int_0^{2\pi} \int_0^{\pi} \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} -\cos \phi \bigg|_{\phi=0}^{\phi=\pi} \, d\theta
\]

\[
= \int_0^{2\pi} 2 \, d\theta = 4\pi
\]

Aside:

Area \( \left( \frac{\text{cylinder}}{2} \right) = 2 \text{ Area} (\bigcirc) + 4\pi = 6\pi \)

\[
\frac{\text{Area} (\square)}{\text{Area} (\bigcirc)} = \frac{3}{2} = \frac{\text{Volume} (\square)}{\text{Area} (\bigcirc)}
\]

This isn't typical, e.g. compare a cube

and a sphere.
With curves, finding

\[
\text{arc length } = \int_C ds = \int_a^b \| c'(t) \| dt
\]

lead to integrating a function \( f: \mathbb{R}^3 \to \mathbb{R} \)

\[
\int_C f(\mathbf{s}) ds = \int_a^b f(c(t)) \| c'(t) \| dt
\]

Similarly, if \( f: \mathbb{R}^3 \to \mathbb{R} \)

\[
\iint_S f(\mathbf{r}(u,v)) \| T_u \times T_v \| \, du \, dv
\]

where \( r: D \to S \) is a parameterization.
Ex: Find the average of \( f(x,y,z) = xy + z \)
over the cone \( \Sigma \) given by \( x^2 + y^2 = z^2 \), \( 0 \leq z \leq 1 \).

Parameterization: \[ \begin{align*}
\Gamma(u,v) &= (v \cos u, v \sin u, v) \\
T_u &= (-v \sin u, v \cos u, 0) \\
T_v &= (\cos u, \sin u, 1) \\
T_u \times T_v &= (v \cos u, v \sin u, -v)
\end{align*} \]

which has length \( \sqrt{2} \sqrt{v} \).

Area: \[ \begin{align*}
\text{Area: } &\iint_S 1 \, dA = \\
&\iint_D \sqrt{2} \sqrt{v} \, du \, dv = \int_0^{2\pi} \int_0^1 \sqrt{2} \sqrt{v} \, du \, dv \\
&= \int_0^1 2\sqrt{2} \pi v \, dv = \sqrt{2} \pi.
\end{align*} \]