Lecture 34: Triple integrals (§6.5)

HW: Due Tuesday April 1

§6.4 7, 13, 18, 29, 30
§6.5 3, 5, 11

Next time: More on §6.5.

Office Hours: None today, Wed @ 4:00-5:30 instead.

Double integrals: \( \iint_{R} f(x, y) \, dx \, dy \rightarrow \iiint_{R} 1 \, dA = \text{Area of } R \)

Average = \( \frac{1}{\text{Area}} \iint_{R} f(x, y) \, dA \)

[Computing total mass, etc.]

Triple integrals: \( R \) region in \( \mathbb{R}^3 \), e.g.

\( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) can discuss

\[ \iiint_{R} f(x, y, z) \, dV \]

\( \iiint 1 \, dV = \text{Volume} \)

Average = \( \frac{1}{\text{Vol.}} \iiint f \, dV \)
For instance, \[
\iiint_{R} 1 \, dV = \text{Volume of } R
\]

Average of \(f\) on \(R\) is \[
\frac{1}{\text{Volume of } R} \iiint_{R} f \, dV.
\]

\[f(x,y,z) = xy + z\]

\[0 \leq x \leq 3, \quad 0 \leq y \leq z, \quad 0 \leq z \leq 1\]

\[\iiint_{R} f \, dV = \int_{0}^{3} \int_{0}^{z} \int_{0}^{1} (xy + z) \, dz \, dx \, dy = 12\]

\[\text{Volume of a unit sphere } x^2 + y^2 + z^2 = 1,\]

plane at height \(z\)

line at fixed y-height

\(y\) runs from \(-\sqrt{1-z^2}\) to \(\sqrt{1-z^2}\)

\(x\) runs from \(-\sqrt{1-y^2-z^2}\) to \(\sqrt{1-y^2-z^2}\)
\[ \iiint_{R} 1 \, dV = \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} 1 \, dx \, dy \, dz \]

Can do with trig substitution or just note that this is the area of the circle of radius \( \sqrt{1-z^2} \).

\[ = \int_{-1}^{1} \pi (1-z^2) \, dz = \pi \left( z - \frac{z^3}{3} \right) \bigg|_{z=-1}^{z=1} \]

\[ = \pi \left( 1 - \frac{1}{3} - (-1 + \frac{1}{3}) \right) = \pi \left( 2 - \frac{2}{3} \right) = \frac{4}{3} \pi. \]

Cf. Archimedes. \[ \text{area } 2\pi \text{ so } \frac{\text{Vol}(\Omega)}{\text{Vol}(\Theta)} = \frac{3}{2}. \]

**Ex:** Consider the region in the pos. octant bounded by the planes \( x = 0, y = 0, z = 2 \) and the surface \( z = x^2 + y^2 \).

Slice by planes of fixed \( z \)-height.
\[ \iiint_{\mathbb{R}} f(x, y, z) \, dy \, dx \, dz = \int_{-2}^{2} \int_{0}^{\sqrt{4 - z^2}} \int_{0}^{\sqrt{z^2 - x^2}} f(x, y, z) \, dy \, dx \, dz \]

**Cylindrical Coordinates:**

\[ (r, \theta, z) \leftrightarrow (r \cos \theta, r \sin \theta, z) \]

\[ 0 \leq r \leq 2 \]

\[ 0 \leq \theta \leq 2\pi \]

**Spherical Coordinates**

\[ 0 \leq \rho \leq 1 \]

\[ 0 \leq \phi \leq \pi \]

\[ 0 \leq \theta \leq 2\pi \]

\[ x = \rho \sin \phi \cos \theta \]

\[ y = \rho \sin \phi \sin \theta \]

\[ z = \rho \cos \phi \]