Lecture 30: More on curl; clitis to multivar. integration.


Next time: §5.1-5.2

Earlier on Math 241:

\[ \vec{F}: \mathbb{R}^3 \to \mathbb{R}^3 \text{ a vector field} \]

\[ \vec{F} = (F_1, F_2, F_3) \]

An associated vector field is

\[ \text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) i - \left( \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) j + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) k \]

\[ \vec{F}: \mathbb{R}^2 \to \mathbb{R}^2 \text{ a vector field given by } (F_1(x,y), F_2(x,y)) \]

If we "promote" to \( \vec{F} = (F_1(x,y), F_2(x,y), 0) \) on \( \mathbb{R}^3 \), then

\[ \text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) k \]

\[ \text{scalar curl} \]

Suppose \( \vec{F}: \mathbb{R}^2 \to \mathbb{R}^2 \) is conservative, \( \vec{F} = \nabla f \), \( f: \mathbb{R}^2 \to \mathbb{R} \)

Then

\[ \Delta \text{curl } \vec{F} = \frac{\partial^2 F_z}{\partial x \partial y} - \frac{\partial^2 F_y}{\partial y \partial x} = \frac{\partial f}{\partial x \partial y} - \frac{\partial f}{\partial y \partial x} = 0. \]
Ex: $F = (y, 0)$ not conservative since path dependent, or since $s.\text{curl } F = 1$.

Q: Is a vector field defined on some region $U$ of $\mathbb{R}^2$

\[ \text{if } s.\text{curl } F = 0, \text{ must } F \text{ be conservative?} \]

A: Yes if $U = \mathbb{R}^2$ or if $U$ "has no holes" [simply connected]

[Will talk about this later in Chapter 8.] But not always

\[ U = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 1 \right\} \]

\[ F(x, y) = \frac{1}{x^2 + y^2} (y, -x) \]

This $s.\text{curl } F = 0$ but is not conservative. [HW.]

[Similar story in $\mathbb{R}^3$ with $\text{curl}(\nabla f) = \partial$.]

Notation: $C$ a curve parametrized by $c: [a, b] \to \mathbb{R}^3$, $F$ a vector field

\[ F = (F_1, F_2, F_3) \]

\[ \int_C F \cdot ds \text{ is sometimes written } \]

\[ \int_C F_1 \, dx + F_2 \, dy + F_3 \, dz, \quad dx = c_1'(t) \, dt, \text{ etc.} \]

[As mentioned, will repeat the curve picture for surfaces in $\mathbb{R}^3$, but first we need learn how to do multi-dimensional integrals...]

\[ , n. \text{ examples...} \]
One var:

\[ \text{Area} = \int_a^b f(x) \, dx \]

computed using fund. thm. of calc.

Two var:

Volume is \( \iiint_R f(x,y) \, dx \, dy \)

Base a square \( R \)

Q1: What does this mean mathematically?

Q2: How do we compute it?

In one var, we addressed the first question as:

Divide \([a,b]\) into segments of length \( \Delta x \).

\( n \) small intervals 

| \( i \)th interval | \( f \) has \( \min M_i \) and \( \max M_i \)
---|---
| \( a \) | \( b \) 

Thus

\[ \sum_{i=1}^{n} M_i \Delta x \leq \int_a^b f(x) \, dx \leq \sum_{i=1}^{n} m_i \Delta x \]

As \( \Delta x \to 0 \) these two bounds converge to the middle.
The two rows:

R:

Each square \( \Delta x \Delta y \) has area \( \Delta x \Delta y \).

Box shown has height

\( = \min \text{ of } f \text{ on sub square} \)

Thus

\[
\sum_{\text{small squares}} (\min \text{ value of } f) \Delta x \Delta y \leq \iint_R f(x,y) \, dx \, dy \leq \\
\sum_{\text{small squares}} (\max \text{ of } f) \Delta x \Delta y
\]

As \( \Delta x, \Delta y \to 0 \), then [provided \( f \) is continuous]

these two bounds converge to define the integral.

Ok, but how do we compute?

Archimedes: (225 B.C.E)

\[
3 : 2
\]

volume \( \pm \) surface area!
Can reduce to one var integrals by cutting into graph of \( f \) slices.

Which have

\[
\text{volume} = A(y) \Delta y
\]

where \( A(y) \) is the area of front of the slice.

To get the total volume, add rolls of slices \( y \) fixed.

\[
\iint f(x,y) \, dx \, dy = \int_c^d \int_c^b \left( \int_a^b f(x,y) \, dx \right) \, dy
\]

\[
= \int_a^b \left( \int_c^d f(x,y) \, dy \right) \, dx
\]