Lecture 15: Derivative Miscellanea

Reminder: Exam in class on Thursday, covering through Section 2.7. Extra office hours: Friday 2-5.

HW: Sect 2.7
* 1, 2, 6, 28, 40

Last time: \( f, g: \mathbb{R}^2 \to \mathbb{R}^2 \) given by
\[
  f(x, y) = (\cos y + x^2, e^x + y) \\
  g(u, v) = (e^{u^2}, u - \sin v)
\]

General Chain Rule: \( g: \mathbb{R}^n \to \mathbb{R}^m, f: \mathbb{R}^m \to \mathbb{R}^k \)
deposeg as differentiable at \( \bar{x}_0 \) and \( f \) is differentiable at \( g(\bar{x}_0) \).
Then \( h = f \circ g \) is differentiable at \( \bar{x}_0 \) with
\[
[Dh(\bar{x}_0)] = [Df(g(\bar{x}_0))][Dg(\bar{x}_0)]
\]

Ex: \( f, g: \mathbb{R}^2 \to \mathbb{R}^2 \) from last time.

\[
[D(f \circ g)(\bar{x})] = [Df(g(\bar{x}))][Dg(\bar{x})]
\]
\[
(2 \ 0) (0 \ 0) = (0 \ 0)
\]
\[
(e \ e)(1 \ -1) = (e \ -e)
\]

Feel free to check directly!
Another way of thinking about

\[ g: \mathbb{R} \rightarrow \mathbb{R}^2 \quad f: \mathbb{R}^2 \rightarrow \mathbb{R} \quad h = f \circ g : \mathbb{R} \rightarrow \mathbb{R} \]

\[ g(t) = (x(t), y(t)) \quad \quad h(t) = f(x(t), y(t)) \]

\[ \frac{dh}{dt} = \left( \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \right) \]

\[ \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \]

x and y are playing two roles.

\[ \frac{dh}{dt} \quad \text{The rate } h \text{ is changing w.r.t. to } t \]

\[ \frac{\partial h}{\partial x} \frac{dx}{dt} + \frac{\partial h}{\partial y} \frac{dy}{dt} \quad \text{The rate } f \text{ changes w.r.t. to } x \text{ times the rate } x \text{ is changing...} \]

Sometimes folks slide the distinction between f and h and write \[ \frac{dh}{dt} = \frac{\partial h}{\partial x} \frac{dx}{dt} + \frac{\partial h}{\partial y} \frac{dy}{dt}. \]

Alternate notations

\[ \frac{\partial f}{\partial x} = f_x = D_x f \quad \frac{\partial f}{\partial y} = f_y = D_y f \]

[Jump now to section 2.7, which has some important topics for min/max.]
Directional Derivatives: \( f : \mathbb{R}^2 \to \mathbb{R} \)

[Already have \( \partial \)-derivatives, measuring change in (x,y) directions.]

But why give the axes special treatment?

Pick \( \vec{x}_0 \in \mathbb{R}^2 \), \( \vec{v} \) a vector.

Then the derivative of \( f \) in direction \( \vec{v} \) at \( \vec{x}_0 \) is

\[
D_{\vec{v}}f = \frac{\text{rate of change in } f \text{ as we move along } \vec{v}}{\text{with } \vec{v}}
\]

Also denoted \( \nabla_{\vec{v}} \)

\[ \frac{d}{dt} \left. f(\vec{x}_0 + t\vec{v}) \right|_{t=0} \]

Ex: \( \vec{v} = (1,0) \) then \( D_{\vec{v}}f(\vec{x}_0) \) is just \( \frac{\partial f}{\partial x}(\vec{x}_0) \)

In general, we find via the chain rule

\( g : \mathbb{R} \to \mathbb{R}^2 \) \( g(t) = (f \circ g)(t) = D(f \circ g)(t) \)

\[
= Df(g(0) = \vec{x}_0) \cdot Dg(0)
\]

\[
= \left( \frac{\partial f}{\partial x}(\vec{x}_0) \quad \frac{\partial f}{\partial y}(\vec{x}_0) \right) \vec{v} = \frac{\partial f}{\partial x}(\vec{x}_0) v_1 + \frac{\partial f}{\partial y}(\vec{x}_0) v_2
\]
Since if \( \vec{x}_o = (x_o, y_o) \), \( \vec{v} = (v_1, v_2) \)
\[ g(t) = (x_o + tv_1, y_o + tv_2) \]
\[ g'(t) = Dg = (v_1, v_2) = \vec{v}. \]

**Gradient:**
\[ \nabla f(\vec{x}_o) = \left( \frac{\partial f}{\partial x}(\vec{x}_o), \frac{\partial f}{\partial y}(\vec{x}_o) \right) \]
\[ \text{grad}\ f \]

So
\[ D_{\vec{v}} f(\vec{x}_o) = \nabla f(\vec{x}_o) \cdot \vec{v} \]

**Note:** \( D_{2\vec{v}} f = \nabla f(\vec{x}_o) \cdot 2\vec{v} = 2(\nabla f(\vec{x}_o) \cdot \vec{v}) = 2D_{\vec{v}} f \)

In what direction is \( f \) increasing most?

\( \vec{u} \) a unit vector, i.e. \( \| \vec{u} \| = 1 \)
\[ D_{\vec{u}} f = \nabla f(\vec{x}_o) \cdot \vec{u} = \| \nabla f(\vec{x}_o) \| \| \vec{u} \| \cos \theta \]
\[ \nabla f \]

Thus the direction of greatest increase is
\[ \frac{\nabla f(\vec{x}_o)}{\| \nabla f(\vec{x}_o) \|} \]

and the rate of increase is \( \| \nabla f(\vec{x}_o) \| \)