Lecture 13: More on derivatives (§2.4 and 2.5)

Last time: Suppose \( f: \mathbb{R}^n \rightarrow \mathbb{R}^m \) is defined near \( \bar{x}_0 \).

The function \( f \) is differentiable near \( \bar{x}_0 \) if there is a linear transformation \( L: \mathbb{R}^n \rightarrow \mathbb{R}^m \) so that

\[
f(\bar{x}_0 + \bar{h}) = f(\bar{x}_0) + L(\bar{h}) + E(\bar{h}) \quad \text{where} \quad \lim_{\bar{h} \to \bar{0}} \frac{E(\bar{h})}{\|\bar{h}\|} = 0.
\]

The matrix of \( L \) is denoted \( \text{D}f(\bar{x}_0) \).

HW: §2.4 44, 62, 64. §2.5 2, 3, 10, 39

Next time: §2.6

Component Functions:

\( \bar{x}: f: \mathbb{R} \rightarrow \mathbb{R}^2 \quad f(x) = (1, 3) \)

\[ f(x) = (f_1(x), f_2(x)) \quad \text{where} \quad f_1, f_2: \mathbb{R} \rightarrow \mathbb{R} \]

In general, \( f: \mathbb{R}^n \rightarrow \mathbb{R}^m \) have \( f(\bar{x}) = (f_1(\bar{x}), f_2(\bar{x}), \ldots, f_m(\bar{x})) \)

where \( f_i: \mathbb{R}^n \rightarrow \mathbb{R} \)

Formula for \( \text{D}f(\bar{x}_0) \) [aka “The Jacobian”]
\[
\mathbf{D}f(\mathbf{x}_0) = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \frac{\partial f_2}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_m}{\partial x_1}(\mathbf{x}_0) \\
\frac{\partial f_1}{\partial x_2}(\mathbf{x}_0) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial f_m}{\partial x_2}(\mathbf{x}_0) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_1}{\partial x_n}(\mathbf{x}_0) & \frac{\partial f_2}{\partial x_n}(\mathbf{x}_0) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}_0)
\end{pmatrix}
\]

How to remember:

\[
\mathbf{D}f(\mathbf{x}_0) \mathbf{h} \approx \text{change in } f \text{ as we move from } \mathbf{x}_0 \text{ to } \mathbf{x}_0 + \mathbf{h} = \begin{pmatrix}
\Delta f_1 \\
\Delta f_2 \\
\vdots \\
\Delta f_m
\end{pmatrix} = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} \\
\frac{\partial f_2}{\partial x_2} \\
\vdots \\
\frac{\partial f_m}{\partial x_m}
\end{pmatrix}
\]

So row i of \(\mathbf{D}f\) should only involve \(f_i\). Take \(\mathbf{h} = \begin{pmatrix} 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix}\) to see column i should only involve \(\frac{\partial}{\partial x_i}\).

Thm: Differentiable functions are continuous

\[
\text{Continuous: } f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \mathbf{E}(\mathbf{h}) \text{ where } \lim_{\mathbf{h} \to \mathbf{0}} \mathbf{E}(\mathbf{h}) = 0
\]

\[
\text{Diff: } f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \mathbf{D}f(\mathbf{x}_0)\mathbf{h} + \mathbf{E}(\mathbf{h}) \text{ where } \lim_{\mathbf{h} \to \mathbf{0}} \frac{\mathbf{E}(\mathbf{h})}{\|\mathbf{h}\|} = 0
\]

As linear transformations are continuous, the 2nd notion is a refinement of the first.
When is a function differentiable?

Need at least that \( \frac{\partial f_i}{\partial x_j} \) all exist at \( x_0 \).

Not enough as consider \( f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \)

This vanishes along both axes so \( \frac{\partial f}{\partial x}(\delta) = \frac{\partial f}{\partial y}(\delta) = 0. \)

But \( f \) isn't continuous at \( \delta \)!

Useful Crit: if \( \frac{\partial f_i}{\partial x_j} \) exist and are continuous near \( x_0 \),
then \( f \) is differentiable at \( x_0 \).

Ex: \( f(x,y) = x^3 y \sin(xy^2) \) is diff on all of \( \mathbb{R}^2 \).

Reason but works is a little complicated (see text)
but is related to:

\[
f(x_0+h, y_0+k) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) h + \frac{\partial f}{\partial y}(x_0+h, y_0) k
\]

Wrong input, but \( \frac{\partial f}{\partial y} \) is cont
so this doesn't matter much.
Ex: Parametric curves

\[ \vec{c} : \mathbb{R} \rightarrow \mathbb{R}^2 \text{ or } \mathbb{R}^3 \]

For instance

\[ \vec{c}(t) = (c_1(t), c_2(t)) = (\cos t, \sin t) \]

\[ \vec{D}\vec{c}(t) = \begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \]

which is basically the velocity vector to the curve. Sometimes denote as \( \vec{c}'(t) \) [and write as a row vector.]

\[ \vec{c}(t) = (\cos t, \sin t, t) \]

\[ \vec{D}\vec{c}(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix} \]

For such a path, \( \vec{c}'(t) \) is velocity and \( \vec{c}''(t) \) is the acceleration. E.g., in the first example

\[ \vec{c}''(t) = \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix} = -\vec{c}'(t) \]

which is why you have centripetal force to exert to keep objects moving in a circle

\[ \vec{F}(t) = m \vec{c}''(t) \]
\( \mathbf{\mathbf{f}} : \mathbb{R}^{2} \rightarrow \mathbb{R}^{2} \quad f(x, y) = (\cos y + x^2, e^{x+y}) \)

\[
Df(x, y) = \begin{pmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}
\end{pmatrix} = \begin{pmatrix} 2x & -\sin y \\
e^{x+y} & e^{x+y} \end{pmatrix}
\]

\( Df(1, 0) = \begin{pmatrix} 2 & 0 \\
e & e \end{pmatrix} \)

\( \mathbf{g} : \mathbb{R}^{2} \rightarrow \mathbb{R}^{2} \quad g(u, v) = (e^{u^2}, u - \sin v) = (g_1, g_2) \)

\[
Dg(u, v) = \begin{pmatrix} 2ue^{u^2} & 0 \\
1 & + \cos v \end{pmatrix} \quad Dg(0, 0)
\]

Consider \( f \circ g : \mathbb{R}^{2} \rightarrow \mathbb{R}^{2} \)

\[
f \circ g(u, v) = f(g(u, v)) = f(e^{u^2}, u - \sin v)
= (\cos(u - \sin v) + e^{2u^2}, e^{u^2} + u - \sin v)
\]

Compute \( D(f \circ g)(\bar{v}) \)...

then has got to be a better way...
Where have you seen this before???