Lecture 12: Derivatives (§2.4)

HW: (Due Feb 12) Section 2.4 # 13, 23, 29, 38

Next time: Rest of §2.4; §2.5.

Reminder: First exam is Thursday, Feb 14.

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Derivatives:

One var: \( f: \mathbb{R} \to \mathbb{R} \)

The tangent line is given by

\[
g(x_0+h) = f(x_0) + f'(x_0)h + \varepsilon(h)
\]

and this "well approximates" \( f \) in the sense that

\[ f(x_0 + h) = f(x_0) + f'(x_0)h + E(h) \]

where \( E(h) \) is small when \( h \) is small.

In particular, \( \lim_{h \to 0} E(h) = 0 \). But that's not enough, consider \( f(x) = |x| \), at \( x_0 = 0 \). Then \( f(h) = f(x_0) + O(h) + E(h) \)

where \( E(h) = |h| \) and \( \lim_{h \to 0} E(h) = 0 \).

[Query: what happens if "\( f'(0) = 1 \)"?]

Correct condition

\[ \lim_{h \to 0} \frac{E(h)}{h} = 0 \]

Similarly,

\[
f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + E(h)
\]

where \( \lim_{h \to 0} \frac{E(h)}{h^2} = 0 \).
Notice that (*) leads to 
\[ \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0) + \frac{E(h)}{h} \]
and so taking \( h \to 0 \) makes the connection to what you know clear.

Suppose \( f : \mathbb{R}^2 \to \mathbb{R} \). The derivative of \( f \) at \( x_0 \) is a linear transformation \( L : \mathbb{R}^2 \to \mathbb{R} \) s.t.

\[ f(x_0 + \bar{h}) = f(x_0) + L(\bar{h}) + E(\bar{h}) \]

where \( L \) is small.

\[ \lim_{h \to 0} \frac{E(h)}{\|h\|} = 0. \]

[Same works for any function \( f : \mathbb{R}^n \to \mathbb{R}^m \).]

\( L \) is given by a matrix of size \([a \ b] \) (a b)

e.g. \( h = (h_1, h_2) \) then

\[ f(x_0 + \bar{h}) = f(x_0) + (a \ b) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + E(\bar{h}) \]

\[ = f(x_0) + ah_1 + bh_2 + E(\bar{h}) \]

To solve for \( a \), let's take \( \bar{h} = (h_1, 0) \)

\[ f(x_0 + (h_1, 0)) = f(x_0) + ah_1 + E(h_1) \]

\[ f(x_0 + h_1, y_0) \quad \text{so} \quad a = \frac{f(x_0 + h_1, y_0) - f(x_0)}{h_1} - \frac{E(h)}{h_1} \]

where \( x_0 = (x_0, y_0) \)
and

\[ a = \lim_{h_1 \to 0} \frac{f(x_0+h_1, y_0) - f(x_0, y_0)}{h_1} \]

which is \( \frac{\partial f}{\partial x} (x_0) \) the partial derivative of \( f \) with respect to \( x \). As \( y_0 \) is fixed this is really just a one-var derivative.

The other entry of the matrix is \( b = \frac{\partial f}{\partial y} (x_0) \).

**Ex:** \( f(x, y) = x^3 y \sin(xy^2) \)

\[ \frac{\partial f}{\partial x} = 3x^2y \sin(xy^2) + x^3y^2 \cos(xy^2) \cdot y^2 \]

\[ \frac{\partial f}{\partial y} = x^3 \sin(xy^2) + x^3y \cos(xy^2) \cdot (2xy) \]

\[ 2xy^2 \cos(xy^2) \]

So we should be able to approximate \( f \) near \( \bar{x}_0 = (1, 1) \) by \( L: \mathbb{R}^2 \to \mathbb{R} \) given by

\[ \begin{pmatrix} 3 \sin(1) + \cos(1) & \sin(1) + 2 \cos(1) \end{pmatrix} \begin{pmatrix} x_0 \ 1.92 \end{pmatrix} \]

When \( f: \mathbb{R}^n \to \mathbb{R}^m \) is defined on a ball near \( \bar{x}_0 \), we say it is differentiable there when we can find a linear map \( L: \mathbb{R}^n \to \mathbb{R}^m \) where
\[ f(x_0 + h) = f(x_0) + L(h) + E(h) \] where \( \lim_{\|h\| \to 0} \frac{E(h)}{\|h\|} = 0 \).

Where \( L \) is denoted \( Df(x_0) \).

[The matrix for \( L \) is given by partial derivatives as \( \nabla f \).

[describe later. For now let's continue to focus on \( \mathbb{R}^2 \to \mathbb{R} \).]

When is \( f \) differentiable at \( x_0 \)? [Query.] Need \( \frac{\partial f}{\partial x}(x_0) \) and \( \frac{\partial f}{\partial y}(x_0) \) to exist.

But that's not enough, e.g. \( f(x,y) = \begin{cases} \frac{xy}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \)

At \( 0 \), both partials are 0, but \( f \) isn't even continuous at \( 0 \) [and thus not even dif].

Useful but: \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) exist and are continuous near \( x_0 \) there \( f \) is differentiable there.

\( \exists x: f(x,y) = x^3y \sin(xy^2) \) is differentiable on all of \( \mathbb{R}^2 \).

Thm: Differentiable functions are continuous.

Continuous means \( f(x_0 + h) = f(x_0) + E(h) \) where \( E(h) \to 0 \) as \( h \to 0 \).

Differentiable means \( f(x_0 + h) = f(x_0) + Df(x_0)h + F(h) \) where \( F(h) \to 0 \) as \( h \to 0 \).

Linear transformations are continuous, the 2nd notion is a refinement of the former.