Lecture 20: Min/Max in action

Last time: \( f: \mathbb{R}^2 \to \mathbb{R} \) with a critical point at \( \vec{x}_0 \)

\[
H = \begin{pmatrix}
\frac{\partial^2 f}{\partial x^2}(\vec{x}_0) & \frac{\partial^2 f}{\partial x \partial y}(\vec{x}_0) \\
\frac{\partial^2 f}{\partial y \partial x}(\vec{x}_0) & \frac{\partial^2 f}{\partial y^2}(\vec{x}_0)
\end{pmatrix}
\]

\( D = \text{det} H \)

a) \( D > 0 \), \( f_{xx}(\vec{x}_0) > 0 \) \( \Rightarrow \) local min

b) \( D > 0 \), \( f_{xx}(\vec{x}_0) < 0 \) \( \Rightarrow \) local max

c) \( D < 0 \) \( \Rightarrow \) saddle

Example: Find distance from the plane \( x - y + 2z = 3 \) to \( \vec{d} \).

\[
z = 3 - x + y
\]

Want to minimize \( f(x,y) = \left( \text{dist from } \left( x,y, \frac{3-x+y}{2} \right) \right)^2 = x^2 + y^2 + \frac{1}{4} \left( 3-x+y \right)^2 \)

[The square is just to make the computations easier.]

Critical Points: \( \nabla f = \vec{0} \) (or undefined)

\[
\frac{\partial f}{\partial x} = 2x - \frac{1}{2} (3-x+y) = \frac{5}{2} x - \frac{1}{2} y - \frac{3}{2}
\]

\[
\frac{\partial f}{\partial y} = 2y + \frac{1}{2} (3-x+y) = -\frac{1}{2} x + \frac{5}{2} y + \frac{3}{2}
\]

Solve \( \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \) \( \iff \)

\[
\begin{align*}
5x - y &= 3 \\
-x + 5y &= -3
\end{align*}
\]

\( \iff \) \( x = \frac{1}{2} \)

\( y = -\frac{1}{2} \)
Since there is only one critical point this must be our minimum, and so the closest point is \((\frac{1}{2}, -\frac{1}{2}, 1)\) at distance \(\sqrt{\frac{3}{2}} \approx 1.2\).

Hey, wouldn't the same reasoning "show" that the maximum distance is also achieved at \((\frac{1}{2}, -\frac{1}{2}, 1)\)?

True, we're missing something. In fact \(f\) has no maximum, as is clear geometrically.

From the picture, it looks like we have a min but clearly we need to think things through more carefully.

One row: May or may not have global extreme values

\[\begin{align*}
\text{no global max} & \quad \text{no global max} \\
\text{or min.} & \quad \text{or min.}
\end{align*}\]

**Extreme Value Theorem:** \(f\) continuous function on \([a, b] = \{a \leq x \leq b\}\). Then \(f\) has a global min and max.
Addendum: These global min/max occur at either a) a critical pt or b) one of the end pts, e.g. 50.63.

Multi Var: D a subset of $\mathbb{R}^2$

Bounded: D is contained inside some ball.

Not enough in 1-var, e.g.

$$ (0,1) = \{0 < x < 1\} $$

$$ f(x) = \frac{1}{x} $$

Closed: D is closed if it contains all its boundary points.

$$ \{ \| x \| < 1 \} $$

$$ \{ \| x \| = 1 \} $$

$$ \{(x,y) \mid 0 < x < 1 \} $$

$$ \{(x,y) \mid 0 \leq x \leq 1 \} $$

$$ \{(x,y) \mid 0 \leq y \leq 1 \} $$
Formally: \( D \) is closed if for each point \( \bar{p} \) not in \( D \), there is an \( r > 0 \) such that \( B_r(\bar{p}) \) misses \( D \).

\[
\begin{array}{c}
\text{vs.} \\
\bar{p} = (0, \frac{1}{2})
\end{array}
\]

Extreme value theorem: Suppose \( D \) is a closed and bounded subset of \( \mathbb{R}^n \). If \( f : D \to \mathbb{R} \) is continuous, then \( f \) has global extrema on \( D \), which occur either at:

a) critical points

b) the boundary of \( D \).

Back to problem at hand.

On \( D \) there's one critic pt

\[
D = \{ ||x|| \leq 2 \}
\]

\( (\frac{1}{2}, -\frac{1}{2}) \) where \( f = \frac{3}{2} \)

On \( 2D \), \( f \geq 4 \). So is the

global min of \( f \) on \( D \).

Outside of \( D \), \( f \geq 4 \) so in fact

\( f \) has a global min on \( \mathbb{R}^2 \) of \( \frac{3}{2} \), achieved

at \( (\frac{1}{2}, -\frac{1}{2}) \). Double check: Apply 2nd der. test