This week: §8.4 \hspace{1cm} \text{Note: ICES online.}

So far, we've a bunch of theorems relating integrals on boundaries with integral of derivatives over regions. These are all the same if we use the language of differential forms:

\[ \omega \text{ a } k \text{-form defined on a region } R \text{ of dimension } k+1. \]

Then:

\[ \int_R \omega = \int_R d\omega \]

The goal of this week is to explain what this means.

Applications: "repackaging" Maxwell's equations; fix the Slepian A.

A \underline{k\text{-form}} is something which can be integrated over a \( k \)-dimensional object. \[ \text{"The bit after the integral sign."} \]

Ex: 0-form: \( f(x) \)

\[ 1 \text{-form: } f(x) \, dx \quad f(x,y,z) \, ds \]

\[ 2 \text{-form: } f(x,y) \, dx \, dy \quad dA \]

\[ [3 \text{-form: } f(x,y,z) \, dx \, dy \, dz \quad dV ] \]
0-forms: A 0-form on \( U \) in \( \mathbb{R}^n \) is just a function \( f: U \to \mathbb{R} \).

1-form: A 1-form on \( U \) in \( \mathbb{R}^3 \) is something of the form

\[
\alpha = f(x, y, z) \, dx + g(x, y, z) \, dy + h(x, y, z) \, dz
\]

where \( f, g, h: U \to \mathbb{R} \) are functions.

Ex: \( \alpha = y^2 \, dx + (x + y) \, dy \), a 1-form on \( \mathbb{R}^2 \)

[Remind how this looks like something]

[you can integrate over a path]

Ex: \( \alpha \) as above

\[
\alpha_{_{(1,2)}} (1, -3) = \begin{pmatrix} 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = 4 - 9 = -5
\]
Thus $\alpha$ assigns to each point in the plane a linear transformation $\mathbb{R}^2 \rightarrow \mathbb{R}$. We can think of this as a "rule" at each point which we use to measure vectors.

We can integrate a 1-form over a curve $C$ like this

$$\int_C \alpha = \int_a^b \alpha_{c(t)}(c'(t)) \, dt$$

$$= \int_a^b f(c(t))c'_1(t) + g(c(t))c'_2(t) \, dt$$

where

$$\alpha = f(x,y) \, dx + g(x,y) \, dy$$

and

$$c(t) = (c_1(t), c_2(t))$$.
As always, this doesn't depend on the parameterization.

\[ C(t) = (\cos t, \sin t) \quad 0 \leq t \leq \pi \]

\[ \alpha = y^2 \, dx + (x+y) \, dy \]

\[ \int \alpha = \int_0^\pi \alpha_{C(t)}(C'(t)) \, dt = \int_0^\pi (\sin^2 t, \cos t + \sin t) \left( \frac{-\sin t}{\cos t} \right) \, dt \]

\[ = \int_0^\pi -\sin^3 t + \cos^2 t + \sin t \cos t \, dt = \]

What's going on here?

1) Before, we needed two things to integrate over a curve: a function and "ds."
   The 1-form \( \alpha \) combines these 2 into one package.

2) A 1-form can be thought of as coming from a vector field \( F \) where

\[ \alpha_p(v) = F(p) \cdot v \quad \text{Then} \quad \int_C \alpha = \int_C F \cdot ds. \]