Monotone Sequences:

Increasing: \( a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq \ldots \)

Decreasing: \( a_1 > a_2 > a_3 > a_4 > a_5 > \ldots \)

A sequence is **bounded** if there is an \( M \) so that \( |a_n| \leq M \) for all \( n \).

**Thm:** A monotone sequence converges if and only if it is bounded.

\[ \{ a_n = n \}_{n=0}^{\infty} \]

\[ \{ \frac{n}{n+1} \}_{n=1}^{\infty} \]

unbounded, divergent

bounded: \( |a_n| \leq 1 \), convergent.
Suppose \( \{a_n\} \) is an increasing sequence with \( |a_n| \leq M \), and thus \(-M \leq a_n \leq M\).

If we take \( L \) to be the smallest number above \( a_n \) for all \( n \), then
\[
\lim_{n \to \infty} a_n = L
\]

That there is such a smallest upper bound is the Completeness Property of the Real Numbers.

Example: \( \{a_n = \frac{2^n}{n!}\}_{n=1}^\infty = \{2, 2, \frac{4}{3}, \frac{4}{15}, \frac{4}{45}, \ldots\} \)

A decreasing sequence, since \( \frac{a_{n+1}}{a_n} = \frac{2}{n+1} \leq 1 \)
Also bounded: \( 0 \leq a_n \leq 9_1 = 2 \), so \(|a_n| \leq 2\).

Hence, it converges. What is the limit? \( a_{12} = 0.00000086 \)
\( a_{20} = 4.31 \times 10^{-13} \)

Indeed,
\[
a_n = \frac{2^n}{n!} = \frac{2(2 \cdot 2 \cdot 2 \cdots 2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdots n} \leq \frac{4}{n} \leq 1
\]

So
\[
0 \leq a_n \leq \frac{4}{n} \implies \lim_{n \to \infty} 0 \leq \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} \frac{4}{n} = 0
\]
and so \( \lim_{n \to \infty} a_n = 0 \).

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### Infinite Series (Sums)

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1
\]
\[
1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}
\]

What does this mean? We only know how to add up finitely many things. [Remind of \( \Sigma \) notation]

\[
\sum_{k=1}^{5} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}
\]

and

\[
\sum_{k=1}^{10} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} = \frac{1023}{1024}
\]
\[ S_1 = \frac{1}{2} \]
\[ S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \]
\[ S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \]
\[ S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} \]

\[ \vdots \]
\[ S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \sum_{k=1}^{n} \frac{1}{2^k} = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n} \]

Now \( \lim_{n \to \infty} S_n = \lim_{n \to \infty} 1 - \frac{1}{2^n} = 1 \) so we say

\[ \sum_{k=1}^{\infty} \frac{1}{2^k} = 1. \]

In general, suppose \( \{a_k\}_{k=1}^{\infty} \) is a sequence and we want to define \( \sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \cdots \).

Let

\[ S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n \]

If \( \lim_{n \to \infty} S_n \) exists and is equal to \( S \), we say that \( \sum_{k=1}^{\infty} a_k \) converges and write \( \sum_{k=1}^{\infty} a_k = S \).