Lecture 12: Sequences (§8.1)

HW #5 Due Wed Oct 1: §7.1 5, 15, 21, 32
§8.1 4, 11, 13, 45

Term in Honors Set.

Next time: More on §8.1

Next week: Infinite Series: \( 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots = \frac{\pi^2}{6} \)

**Sequences:** An infinite list of numbers.

- First term
  \( \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots, \frac{1}{n}, \ldots \} \)
- Second term
  \( \{1, -1, 1, -1, 1, -1, 1, -1, \ldots \} \)
- Third term
  \( \{1, 3, 5, 7, 9, 11, 13, \ldots \} \)
- Fourth term
  \( \{3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, \ldots \} \)

Individual numbers are called terms, often denoted \( \{a_1, a_2, a_3, a_4, \ldots \} \). Usually specified by a rule: \( a_n = \frac{1}{n} \) for \( n = 1, 2, 3, \ldots \)

Write \( \sum_{n=1}^{\infty} = \left\{ \frac{1}{n} \right\}_{n=1}^{\infty} \).

\( a_1 = 1 \quad a_2 = \frac{1}{2} \quad a_3 = \frac{1}{3} \quad a_4 = \frac{1}{4} \)

Examples: \( \left\{ -1^n \right\}_{n=0}^{\infty} \), \( \left\{ 2n+1 \right\}_{n=0}^{\infty} \), \( \left\{ m^{\text{th}} \text{ digit of } \pi \right\}_{n=1}^{\infty} \), \( \{An\} \) from honors set.
Limits: \( a_n = \frac{1}{n} \) for \( n = 1, 2, 3, \ldots \): \( 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots \).

Clearly \( \lim_{n \to \infty} a_n = 0 \), but what does this mean precisely?

Idea: \( a_n \) is very close to 0 if \( n \) is large enough.

Definition: A sequence \( \{a_n\}_{n=n_0}^{\infty} \) converges to \( L \) (i.e., \( \lim_{n \to \infty} a_n = L \)) if for every \( \varepsilon > 0 \) there is an \( N \) so that

\[
|a_n - L| < \varepsilon \quad \text{for every} \quad n > N.
\]

As shown, take \( N = 3 \).
Claim: \( \lim_{n \to \infty} \frac{1}{n^2} = 0 \)

Warm Up: Challenge/Response

Given \( \varepsilon = \frac{1}{5} \), I'll take \( N = 4 \). Now for \( n > N = 4 \) we have

\[
|a_n - L| = \left| \frac{1}{n^2} \right| \leq \frac{1}{16} < \frac{1}{5} = \varepsilon.
\]

[Repeat a couple times]

Proof: Let \( \varepsilon > 0 \) be given. Choose an integer \( N \) where \( N > \sqrt{\frac{\varepsilon}{\varepsilon}} \). Then for \( n > N \) we have

\[
|a_n - L| = \left| \frac{1}{n^2} - 0 \right| = \frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon
\]

Since \( N > \sqrt{\frac{\varepsilon}{\varepsilon}} \) \( \Rightarrow \) \( N^2 > \frac{1}{\varepsilon} \) \( \Rightarrow \) \( \varepsilon > \frac{1}{N^2} \).

Can also think about numerically:

\[
\begin{array}{l|l}
\text{an} & \\
1 & 1.00000000 \\
2 & 0.25000000 \\
3 & 0.11111111 \\
4 & 0.06250000 \\
5 & 0.04000000 \\
6 & 0.02777777 \\
7 & 0.01000000 \\
8 & 0.00000000 \\
9 & 0.00000000 \\
10 & 0.00000000 \\
\end{array}
\]
In fact, the notion of limit is (roughly) equivalent to say that for each m, the first m digits of the numbers in the table stop changing after some point. 

\[ \text{Ex: } a_n = \cos^{-1}(-1 + \frac{1}{2 \cdot 3^n}) \]

**Warning:** Just looking at a finite table of numbers is not a proof. But it is good evidence...

**Divergent Examples:**

1) \[
\lim_{n \to \infty} \frac{n^2 + 1}{2n + 3} = \lim_{n \to \infty} \frac{n^2 + 1}{2n + 3} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} \frac{n + \frac{1}{n}}{2 + \frac{3}{n}} = \infty
\]

So \[ \sum_{n=1}^{\infty} \frac{n^2 + 1}{2n + 3} \] diverges.

2) \[ \{1, -1, 1, -1, 1, -1, \ldots \} \] also diverges.

**Proof:** (only if asked)

Suppose \( \{a_n = (-1)^n\} \) converges to \( L \).

Then for \( \varepsilon = 1/2 \), there is a \( N \) s.t. \( |a_n - L| < \frac{1}{2} \) for all \( n \geq N \).

But \( a_N, a_{N+1} = 1, -1 \) in some order, and no \( L \) is \( 1/2 \) from both of these, a contradiction.