Improper integrals: Those where the Fundamental Theorem of Calculus does not directly apply.

\[ \int_0^\infty e^{-x} \, dx = \lim_{R \to \infty} \int_0^R e^{-x} \, dx = \lim_{R \to \infty} \left[ -e^{-x} \right]_0^R = \lim_{R \to \infty} -e^{-R} + e^0 = 1 \]

Can also look at integrals over the whole x-axis:

\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \]
Q: How should we define \( \int_{-\infty}^{\infty} f(x) \, dx \)?

Try 1: \( \lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx \)

Not quite right: \( \int_{-\infty}^{\infty} x \, dx = \lim_{R \to \infty} \int_{-R}^{R} x \, dx \)

\[ = \lim_{R \to \infty} \left. \frac{x^2}{2} \right|_{-R}^{R} = \lim_{R \to \infty} 0 = 0. \]

This isn't unreasonable, but what about \( \int_{-\infty}^{\infty} x+1 \, dx = \lim_{R \to \infty} \int_{-R}^{R} x+1 \, dx \) ??

Some graph as \( x \), but shifted over to the left,

so answer which does not exist.

Correct: If \( f \) is cont on \((-\infty, \infty)\), we write

\[ \int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{a} f(x) \, dx + \int_{a}^{\infty} f(x) \, dx \]

for any constant \( a \),

and say \( \int_{-\infty}^{\infty} f(x) \, dx \) converges if and only if both of the Riemann integrals converge.
\[ \int_{-\infty}^{\infty} x \, dx = \int_{-\infty}^{0} x \, dx + \int_{0}^{\infty} x \, dx \]

diverges as \( \int_{-\infty}^{0} x \, dx \) does.

Comparison Test: Deciding when \( \int_{a}^{\infty} f(x) \, dx \)
converges, even when we can't explicitly compute it.

\[ \int_{0}^{\infty} e^{-x^2} \, dx, \text{ where we don't have a closed } \]
form for \( \int e^{-x^2} \, dx \)

General method: \( f, g \) cont,

\[ 0 \leq f(x) \leq g(x) \]

for \( x \) in \([a, \infty)\)

Then if \( \int_{a}^{\infty} g(x) \, dx \) converges, so does \( \int_{a}^{\infty} f(x) \, dx \)

clear: \( \text{for a function } h(x) \geq 0, \text{ then } \)

finite area \( \iff \int_{a}^{\infty} h \, dx \) converges

infinite area \( \iff \int_{a}^{\infty} h \, dx \) diverges

means "if and only if"
As the graph of \( f \) lies below that of \( g \), clearly the area under \( f \) is \( \leq \) the area under \( g \). So if the area under \( g \) is finite, so is the area under \( f \).

Similarly: If \( \int_a^\infty f(x) \, dx \) diverges, so does \( \int_a^\infty g(x) \, dx \).

Ex: \( \int_2^\infty e^{-x^2} \, dx \)  On \([2, \infty)\) we have \( x \leq x^2 \), so \( e^x \leq e^{x^2} \) so \( e^{-x^2} \leq e^{-x} \).

As \( \int_1^\infty e^{-x} \, dx \) converges, so does \( \int_1^\infty e^{-x^2} \, dx \).

[Can also apply to \( \int_0^\infty e^{-x^2} \, dx \) by noting that this is equal to \( \int_2^\infty e^{-x^2} \, dx + \int_2^{\infty} e^{-x^2} \, dx \).]
Exam: \[ \int_{1}^{\infty} \frac{3 + \cos x}{x} \, dx \]

Now \(-1 \leq \cos x \leq 1\), so

\[ \frac{2}{x} \leq \frac{3 + \cos x}{x} \leq \frac{4}{x} \]

As \( \int_{1}^{\infty} \frac{2}{x} \, dx \) diverges, so does \( \int_{1}^{\infty} \frac{3 + \cos x}{x} \, dx \).