Lecture 18: Alternating Series (§8.4) / General Series (§8.5)

HW #7: Oct 15: §8.4 #: 5, 6, 19, 24, 29, 37

Next time: Move on §8.5.

Last time: Alternating Series: \( \sum_{k=0}^{\infty} (-1)^k a_k = a_0 - a_1 + a_2 - a_3 + \ldots \)
where \( a_k \geq 0 \).

Ex: \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots = \ln 2 \)

Partial Sums:

\( S_1 = 1 \)
\( S_2 = S_1 - \frac{1}{2} = \frac{1}{2} = 0.5 \)
\( S_3 = S_2 + \frac{1}{3} = \frac{5}{6} = 0.833... \)
\( S_4 = S_3 - \frac{1}{4} = 0.5833... \)
\( S_5 = S_4 + \frac{1}{5} = \frac{47}{60} = 0.7833... \)
\( S_6 = S_5 - \frac{1}{6} = 0.6166... \)

[Hand-drawn]

Series converges to \( \ln 2 = 0.6931... \)

Alternating Series Test: Suppose \( 0 < a_{k+1} \leq a_k \) for all \( k \geq 0 \), and \( \lim_{k \to \infty} a_k = 0 \). Then \( \sum_{k=0}^{\infty} (-1)^k a_k \) converges.
Idea: \( \sum_{k=0}^{\infty} (-1)^k a_k = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \ldots \)

\[ S_n = \sum_{k=0}^{n} (-1)^k a_k \]

Point: \( a_{k+1} \leq a_k \)

\[ S_1 \quad S_3 \quad S_5 \quad S_4 \quad S_2 \quad S_0 = a_0 \]

Notice: \( S_1 \leq S_3 \leq S_5 \leq S_7 \leq \ldots \) \( \{ S_{2j+1} \}_{j=0}^{\infty} \) — increasing

\( S_0 \geq S_2 \geq S_4 \geq S_6 \geq \ldots \) \( \{ S_{2j} \}_{j=0}^{\infty} \) — decreasing

Also \( S_1 \leq S_n \leq S_0 \) for all \( n \). As bounded monotone sequences converge, \( \lim_{j \to \infty} S_{2j+1} = \text{Lodd} \)

\[ S_1 \quad S_3 \quad S_5 \quad \ldots \quad S_{2j+1} \quad S_{2j} \quad S_1 \quad S_3 \quad S_5 \]

\( \lim_{j \to \infty} S_{2j} = \text{Leven} \)

Now

\[ 0 = \lim_{j \to \infty} a_{2j+1} = \lim_{j \to \infty} S_{2j+1} - S_{2j} = \lim_{j \to \infty} S_{2j+1} - \lim_{j \to \infty} S_{2j} = \text{Lodd} - \text{Leven} \]

Thus \( \text{Lodd} = \text{Leven} \) and \( \lim_{n \to \infty} S_n \) exists. So \( \sum_{k=1}^{\infty} a_k \) converges.
Error bounds: Suppose the Alt. series test says that
\[ \sum_{k=0}^{\infty} (-1)^k a_k \] converges. Then
\[ \left| \sum_{k=0}^{n} (-1)^k a_k - \sum_{k=1}^{\infty} (-1)^k a_k \right| \leq a_{n+1} \]

remainder \( R_n \)

\[ \sum_{k=1}^{10000} (-1)^k k = 0.6931971... \]

is within \( a_{10001} = \frac{1}{10001} \) of
\[ \sum_{k=1}^{\infty} (-1)^k k = 0.6931471... \]

General series: (§8.5) [Recap story so far.]

\[ \sum_{k=1}^{\infty} \frac{\cos k}{k^2} = \cos 1 + \frac{1}{4} \cos 2 + \frac{1}{9} \cos 3 + \frac{1}{16} \cos 4 + \frac{1}{25} \cos 5 + \ldots \]

\[ > 0 \quad < 0 \quad > 0 \]

Absolute Convergence:
\[ \sum_{k=1}^{\infty} a_k \] converges absolutely if \[ \sum_{k=1}^{\infty} |a_k| \] converges.

Point: If \[ \sum_{k=1}^{\infty} a_k \] is absolutely convergent, then it converges.

[Give rough idea of why, careful explanation for next time. Useful as we now have many tools for dealing with pos. series.]
Ex: \[ \sum_{k=1}^{\infty} \frac{\cos k}{k^2} \]

Consider \[ \sum_{k=1}^{\infty} \left| \frac{\cos k}{k^2} \right| \].

Now this converges, since \( \left| \frac{\cos k}{k^2} \right| \leq \frac{1}{k^2} \) and \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) converges.

So it is absolutely convergent, hence converges.

\[ \sum_{k=1}^{\infty} \frac{\cos k}{k^2} = 0.324... \quad \text{vs.} \quad \sum_{k=1}^{\infty} \left| \frac{\cos k}{k^2} \right| = 0.927... \]

Ex: \[ \sum_{k=1}^{8} \frac{(-1)^{k+1}}{k} \]

converges, but is not absolutely convergent since \( \sum_{k=1}^{\infty} \frac{1}{k} \) diverges.

Why absolute convergence implies convergence:

\[ \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left| a_k \right| - \sum_{k=1}^{\infty} \left| a_k \right| \]

\[ = \sum_{k=1}^{\infty} \left| a_k + \left| a_k \right| - \left| a_k \right| \right| \]

\[ = \sum_{k=1}^{\infty} \left| a_k + \left| a_k \right| - \left| a_k \right| \right| \]

\[ \geq 0 \]

converges by assumption.

\[ \implies \exists \sum_{k=1}^{\infty} a_k \text{ converges} \]

\[ \sum_{k=1}^{\infty} 2 \left| a_k \right| \] which converges as \( \sum_{k=1}^{\infty} \left| a_k \right| \) does.