Lecture 25: Taylor Series (§8.7)

HW #8 (Oct 29): §8.7: #15

Next time: Finish §8.7, start 8.8.

Last time: $f$ infinitely differentiable at $c$.

Taylor Series: $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k$

Q1: What is the radius of convergence $r$ of $f$?

If $r > 0$ then $g(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k$ gives a function on $(c-r, c+r)$ with $g^{(k)}(c) = f^{(k)}(c)$ for all $k$.

Q2: Does $f(x) = g(x)$ on $(c-r, c+r)$?

[Have an example where answers are (yes, no):]

When the answer to Q2 is yes, the Taylor Polynomials

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} x^k$$

give approximations to $f(x)$.

[In practice one usually works with these.]
Ex: \( f(x) = e^x \) Taylor series about 0: \[ \sum_{k=0}^{\infty} \frac{1}{k!} x^k \text{ and } \lim_{n \to \infty} = 0 \]

\[ P_1(x) = 1 + x \]
\[ P_2(x) = 1 + x + \frac{1}{2} x^2 \]
\[ P_3(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 \]
\[ P_4(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 \]

Can't tell \( P_4 \) from \( e^x \) on this (horizontal) scale except at the very end.

Note: Each \( P_n \) only approximates \( e^x \) "near" 0.
\[
0 \leq R_n(\frac{1}{2}) \leq \frac{e^{\frac{1}{2}}}{8} \leq \frac{e^{\frac{1}{2}}}{8} \leq \frac{4^{\frac{1}{2}}}{8} = \frac{1}{4}
\]

Thus \( e^{\frac{1}{2}} \) is in \([3^{\frac{1}{2}}, 3^{\frac{1}{2}} + \frac{1}{4}] = [1.5, 1.75]\).

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For each \( x \), the series \( \sum_{k=0}^{\infty} \frac{1}{k!} x^k \) converges to \( e^x \).

**Reason:**
Fix \( x \). There is an \( M \) so that \( |e^x| \leq M \) for all \( x \) between \( 0 \) and \( x \) [since \( e^x \) is continuous].

Then

\[
|P_n(x) - e^x| = \left| e^{\frac{x}{2}} x^{n+1} \right| \leq \frac{M}{(n+1)!} |x|^{n+1},
\]

which goes to 0 as \( n \to \infty \). So \( \sum_{k=0}^{\infty} \frac{1}{k!} x^k = e^x \)

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**Ex:** \( e = \sum_{k=0}^{\infty} \frac{1}{k!} \) : use this to estimate \( e \) to \( 10^{-10} \)

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**Note:** \( R_n(1) = \frac{e^{\frac{1}{2}}}{(n+1)!} \) for \( 0 \leq z \leq 1 \). So \( |R_n(1)| \leq \frac{3}{(n+1)!} \)

In particular, if \( n = 13 \), then \(|R_n(1)| < 3.4 \times 10^{-11}\)

So \( e \approx 2.7182818284\ldots \)
Taylor's Theorem: Suppose \( f \) has \((n+1)\) derivatives on \((-r, +r)\) for some \( r > 0 \).

Consider \( P_n = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x-c)^k \). Then for \( x \) in \((-r, +r)\)
the error

\[
R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} (x-c)^{n+1}
\]

where \( z \) is between \( x \) and \( c \).

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**Ex:** \( f(x) = e^x \) \( P_1 = 1 + x \)

At \( x = \frac{1}{2} \)

\[
P_1\left(\frac{1}{2}\right) = \frac{3}{2} = 1.5
\]

\[
f\left(\frac{1}{2}\right) = e^{\frac{1}{2}} = 1.6487...
\]

So Taylor's Theorem says:

\[
R_1\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) - P_1\left(\frac{1}{2}\right) = 0.1487... = \frac{f^{(2)}(z)}{2} \left(\frac{1}{2}\right)^2 = \frac{e^z}{8}
\]

for some \( z \) between 0 and \( \frac{1}{2} \). (Turns out \( z = 0.17376... \))

Why useful? Well, \( e^x \) is increasing, so even without knowing what \( z \) is we get
Brownian Motion: Dust moving in sunlight

Brown (19th century): pollen grains on surface of water. Picked up by Einstein (1905)

2000 years earlier, the roman Lucretius used this as an argument for the existence of molecules.