Lecture 20: Conditionally Convergent Series (8.5) / Power Series (8.6)

Exam Friday: Bring sheet Covers: 8.1-8.5 (not 7.1!)

Review Problems:

Chapter 8 review: True-False: #1-11
Ex: 1-52 except 15.
Also 8.5 #1-38
Review Wed: Email me topics (mmendil@illinois.edu)

Conditionally Convergent: \( \sum_{k=1}^{\infty} a_k \) converges but \( \sum_{k=1}^{\infty} |a_k| \) diverges

\[
\text{Ex: } \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots = \ln 2
\]

Fun Fact: The terms of a conditionally convergent series can be rearranged so they sum to any number, e.g. \( \pi \) or 17. On rearranged so as to diverge.

With an absolutely convergent series, the order of the terms doesn't matter. Will renumber the alt. harmonic series so it diverges.

Positive Terms: \( 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots = \sum_{k=0}^{\infty} \frac{1}{2k+1} \)

Diverges by the integral test (or L.C.T with \( \frac{1}{k} \))
Break into pieces with sum $> 1$:

$$
1 + \left(1 + \frac{1}{3} + \cdots + \frac{1}{29}\right) + \left(\frac{1}{31} + \cdots + \frac{1}{221}\right) + \left(\frac{1}{223} + \cdots + \frac{1}{1649}\right) + \left(\frac{1}{1651} + \cdots \right) + \frac{1}{1.0025} + \frac{1}{1.0006} + \frac{1}{1.0029}.
$$

Now insert a negative term between each block:

$$
1 - \frac{1}{2} + \left(\frac{1}{3} + \cdots + \frac{1}{29}\right) - \frac{1}{4} + \left(\frac{1}{31} + \cdots + \frac{1}{221}\right) - \frac{1}{6} + \left(\frac{1}{223} + \cdots + \frac{1}{1649}\right) - \frac{1}{8} + \cdots
$$

This diverges to $\infty$.

So we've rearranged the alternating harmonic series into a divergent series. [Limit does with an absolutely convergent series.]

**Example:**

$$
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots
$$

where the pos terms sum to about 1.233

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**Power Series (§ 8.6)**

$$
\sum_{k=0}^{\infty} \frac{1}{k!}\cdot x^k = 1 + x + \frac{1}{2}\cdot x^2 + \frac{1}{6}\cdot x^3 + \frac{1}{24}\cdot x^4 + \cdots
$$
Here \( x \) is a variable for which we can plug in specific values. E.g. take \( x = 2 \), and \( \sum_{k=0}^{\infty} \frac{1}{k!} 2^k = 7.38905 \)

This particular power series converges for any fixed \( x \) by the ratio test:

\[
\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \frac{|x|^{k+1}}{(k+1)!} \frac{k!}{|x|^k} = \lim_{k \to \infty} \frac{|x|}{k+1} = 0.
\]

Thus, the power series defines a function \( f(x) = \sqrt{2} e^x \) in fact

\[
\sum_{k=0}^{\infty} \frac{1}{k!} x^k = e^x
\]

Many functions can be expressed this way:

\[
\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 - \frac{1}{720} x^6 + \cdots
\]

\[
\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x-1)^k = (x-1) - \frac{1}{2} (x-1)^2 + \frac{1}{3} (x-1)^3 - \cdots
\]

Why?

Useful: Find \( \cos 1 \) to within 0.0001 = \( \frac{1}{10,000} \)

\[
\cos 1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} + \frac{1}{40320} - \cdots
\]

By alternating series test, the error \( S_n \) and \( \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \) is at most \( |a_{n+1}| \). Thus \( 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} = \frac{389}{720} = 0.5402777 \ldots \) is within \( \frac{1}{40320} \) of \( \cos 1 = 0.540302305 \ldots \)
Natural: Power series come out of looking at polynomial approximations of functions.

Equation of tangent line \( y = x + 1 \) \( \text{as } \left( \frac{d}{dx} e^x \right) \bigg|_{x=0} = 1 \)

\( P_1 = 1 + x \)

\( P_2 = 1 + x + \frac{1}{2} x^2 \) "best fit parabola"

\( P_3 = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 \) "best fit cubic"

\( \left( \frac{d}{dx^k} P_n(x) \right) \bigg|_{x=0} = \left. \frac{d}{dx^k} e^x \right|_{x=0} = 1 \)