Lecture 27: Fourier Series (§8.9)

HW (Nov 5): §8.9: #1

Next Due: (§9.1)

Fourier Series:
\[
\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \sin kx + b_k \cos kx)
\]
used to describe functions that are periodic of
period $2\pi$; $f(x + 2\pi) = f(x)$

\[3 \sin x + \cos 2x = f(x)\]

Put up before hand.
Q: Which functions have Fourier series, and how do we find the coefficients?

Suppose \( f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \).

Then
\[
\int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \, dx + \sum_{k=1}^{\infty} \left( a_k \int_{-\pi}^{\pi} \cos kx \, dx + b_k \int_{-\pi}^{\pi} \sin kx \, dx \right)
\]
\[
= a_0 \pi + \sum_{k=1}^{\infty} (0 + 0) = a_0 \pi.
\]

So
\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx
\]

Also
\[
\int_{-\pi}^{\pi} f(x) \cos nx \, dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \cos nx \, dx + \sum_{k=1}^{\infty} \left( a_k \int_{-\pi}^{\pi} \cos kx \cos nx \, dx + b_k \int_{-\pi}^{\pi} \sin kx \cos nx \, dx \right)
\]
By old HW we know all these integrals are zero except \( \int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi \). Thus

\[
\int_{-\pi}^{\pi} f(x) \cos nx \, dx = a_n \pi
\]

Thus and similarly

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx
\]

Ex: Square wave: \( f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \\ \text{Periodic elsewhere} \end{cases} \)

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \left( \int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} 1 \, dx \right) = 1
\]
\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \left( \int_{-\pi}^{0} \cos nx \, dx + \int_{0}^{\pi} \cos nx \, dx \right) \]

\[ = \frac{1}{\pi} \left( \frac{1}{n} \sin nx \bigg|_{x=\pi}^{x=0} \right) = 0 \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{0}^{\pi} \sin nx \, dx \]

\[ = \frac{1}{\pi n} (-\cos nx) \bigg|_{x=0}^{x=\pi} = \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi n} & n \text{ odd} \end{cases} \]

So the Fourier series is

\[ \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} \sin ((2k+1)x) \]

Does this converge to our original \( f \)? E.g. consider

\[ F_n(x) = \frac{1}{2} + \sum_{k=0}^{n} \frac{2}{(2k+1)\pi} \sin ((2k+1)x) \]
In fact, \( F_h(n) \to f(x) \) as \( n \to \infty \) except for \( x = k\pi \) for some integer \( k \), where it converges to \( \frac{1}{2} \).

**Fourier Convergence Theorem:** If \( f \) is a \( 2\pi \)-periodic function, and if \( f \) is continuous except for finitely many jumps, then its Fourier series converges to \( f \) except at the jumps.
Other periods: if \( f \) has period \( T = 2L \), then use

\[
\frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos \frac{k\pi x}{L} + b_k \sin \frac{k\pi x}{L} \right)
\]

where

\[
a_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos k\pi x \, dx
\]

\[
b_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin k\pi x \, dx
\]

Wavelets, etc.: JPEG 2000.