Lecture 33: Properties of curves in polar coordinates (§9.5)

HW: §9.5 # 9, 13, 15, 20, 39

Next time: §9.6

Last time:

Polar coordinates:

Plane curves:

\[ r = 2 \]

\[ r = 2 \sin \theta \]

\[ r = \sin 3\theta \]

Tangent lines:

Ex: Find the slope of the tangent line to the curve \( r = \sin 3\theta \) at \( \theta = \frac{\pi}{6} \)

\[ x(\theta) = r \cos \theta = \sin 3\theta \cos \theta \]

\[ y(\theta) = r \sin \theta = \sin 3\theta \sin \theta \]
\[ \text{slope} = \frac{y'(\pi/6)}{x'(\pi/6)} = \frac{3 \cos 3\theta \sin \theta + \sin 3\theta \cos \theta}{3 \cos 3\theta \cos \theta - \sin 3\theta \sin \theta} \left|_{\theta = \pi/6} \right. \]

\[= \frac{\cos \pi/6}{\sin \pi/6} = \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} = \sqrt{3} \]

Area:

\[r = f(\theta)\]

\[\theta_{i+1} - \theta_i = \Delta \theta\]

Area of \(i\)th wedge \(\approx\) Area of circular wedge of rad \(f(\theta_i)\) angle \(\Delta \theta\) = \(\frac{f(\theta_i)^2}{2} \Delta \theta\)

Wedge of circle of radius \(f(\theta_i)\)
Area = \sum_{i=1}^{n-1} \text{Area of } i^{th} \text{ wedge} \approx \sum_{i=0}^{n-1} \frac{f(\theta_i)^2}{2} \Delta \theta \approx \int_0^{2\pi} \frac{f(\theta)^2}{2} d\theta

So as \Delta \theta \to 0

\text{Area} = \int_0^{2\pi} \frac{f(\theta)^2}{2} d\theta

\text{Ex:}

r = \sin 3\theta

Find area inside eleven leaf.

Note: Need to be careful; doing the each loop only takes twice \pi/3

\text{Area} = 3 \left( \text{Area of one leaf} \right) = 3 \int_0^{\pi/3} \frac{1}{2} \sin^2 3\theta \, d\theta = 3 \int_0^{\pi/3} \frac{1}{4} (1 - \cos 6\theta) \, d\theta

= \frac{3}{4} \left( \theta - \frac{1}{6} \sin 6\theta \right) \bigg|_0^{\pi/3} = \frac{\pi}{4}
Arc Length:  \[ \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \, dt \]

Suppose we have a polar curve given by

\[ r = f(\theta) \].

Then arc-length formula becomes

\[ \int_a^b \sqrt{(f'(\theta))^2 + (f(\theta))^2} \, d\theta \]

Since

\[ x(\theta) = r \cos \theta = f(\theta) \cos \theta \]
\[ y(\theta) = r \sin \theta = f(\theta) \sin \theta \]

and so \((x'(t))^2 + (y'(t))^2 = (f'(\theta) \cos \theta - f(\theta) \sin \theta)^2 + (f'(\theta) \sin \theta + f(\theta) \cos \theta)^2 \) \[ = (f'(\theta))^2 \cos^2 \theta + (f'(\theta))^2 \sin^2 \theta + (f'(\theta))^2 \sin \theta + (f(\theta))^2 \cos^2 \theta \]
\[ = (f'(\theta))^2 + (f(\theta))^2 \].