Lecture 30: Properties of Plane Curves

HW: (Nov 12): 89.2 #21, 26  89.3 #5, 7.

Midterm Friday, send me topics for Wed. Review on Sec. 8.6-9.3 (through pg 737)

Extra Office Hours: Wed 3-5:00  Thur 9-11

Tangent Lines:

\[
\text{slope of tangent line at } t_0 = \frac{dy}{dx}(t_0) = \frac{y'(t_0)}{x'(t_0)}
\]

If both \( y'(t_0) \) and \( x'(t_0) \) are 0, typically no tangent line.

\( \text{Ex: } x = t^2, \quad y = t^3 \)

\[
x'(t) = 2t \bigg|_{t = 0} = 0 \quad y'(t) = 3t^2 \bigg|_{t = 0} = 0
\]

Notes: This curve is also given by \( x^3 - y^2 = 0 \)

If \( x > 0 \) we have \( t = \sqrt{x} \) and \( y = x^{3/2} \), which explains the graph.
Suppose \( x(t), y(t) \) for \( a \leq t \leq b \) trace out a closed curve once (i.e. curve does not intersect itself except for \( (x(a), y(a)) = (x(b), y(b)) \))

**Clockwise:**
\[
\text{Area} = \int_a^b y(t) x'(t) \, dt = -\int_a^b x(t) y'(t) \, dt
\]

**Counterclockwise:**
\[
\text{Area} = -\int_a^b y(t) x'(t) \, dt = \int_a^b x(t) y'(t) \, dt
\]

[Why does this work? Green's Theorem from Vector Calculus ...
Final! That's calculus in higher dimensions...]

**Ex:**
\[x(t) = 2 \cos t \]
\[y(t) = 2 \sin t \]
\[0 \leq t \leq 2\pi \]
\[y'(t) = 2 \cos t \]

\[
\text{Area} = \int_0^{2\pi} x(t) y'(t) \, dt = 4 \int_0^{2\pi} \cos^2 t \, dt =
\]
\[= 4 \int_0^{2\pi} \frac{1}{2} (1 + \cos 2t) \, dt = 4 \pi = \pi (2)^2 \text{ as expected.} \]
Length of a curve:

\[ \text{(dist)} = (\text{rate})(\text{time}) \implies V(t) = \text{Velocity at time } t = \frac{d}{dt}(\text{dist}) \]

Suppose curve is parametrized by \((x(t), y(t))\). What is \(V(t)\)?

\[ V(t) = \sqrt{(x'(t))^2 + (y'(t))^2} \]

So:

Length of curve \( = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \, dt \)

Ex:

\[ x(t) = 2 \cos t \quad \text{for } 0 \leq t \leq 2\pi \]
\[ y(t) = 2 \sin t \]

\[ V(t) = \sqrt{(2 \sin t)^2 + (2 \cos t)^2} = 2 \]

\[ \text{Length} = \int_0^{2\pi} V(t) \, dt = \int_0^{2\pi} 2 \, dt = 4\pi \]

\[ = 2\pi (2) \text{ as expected.} \]
Example: \( X = t^2 - 1 \)
\( y = t \)
\( 0 \leq t \leq -1 \)

\[
\text{Length} = \int_{0}^{1} \sqrt{(X'(t))^2 + (y'(t))^2} \, dt = \int_{0}^{1} \sqrt{(2t)^2 + 1} \, dt
\]

Now:
\[
\int \sqrt{(2t)^2 + 1} \, dt = \int \frac{1}{2} \sec^3 \theta \, d\theta = \text{reduction formula}
\]

\[
\frac{1}{2} \left( \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \int \sec \theta \, d\theta \right)
\]

\[
= \frac{1}{4} \sec \theta \tan \theta + \frac{1}{4} \ln |\sec \theta + \tan \theta| + C
\]

\[
= \frac{1}{2} \sqrt{2t^2 + 1} + \frac{1}{4} \ln |\sqrt{2t^2 + 1} + 2t| + C
\]

So:
\[
\text{Length} = \left. \frac{1}{2} \sqrt{2t^2 + 1} + \frac{1}{4} \ln |\sqrt{2t^2 + 1} + 2t| \right|_{0}^{1} = \frac{1}{2} \sqrt{5} + \frac{1}{4} \ln (\sqrt{5} + 2) - 0
\]

\[
= \frac{1}{2} \sqrt{5} + \frac{1}{4} \ln (\sqrt{5} + 2) \approx 1.478942...
\]

Compare to diagonal path \( \sqrt{2} \approx 1.4142... \)
Cycloid:

After time $t$, how much has the light rotated?

**Ans:** $\theta = t$

So $x(t) = t + \sin t$

$y(t) = 1 + \cos t$