1. Let $M$ be a compact surface with a metric of constant $-1$ curvature. Prove that there exists a unique geodesic in each free homotopy class of loops in $M$.

2. Let $M$ be a compact surface with $\chi(M) < 0$. If $M$ is non-orientable, prove that $M$ has a metric of constant negative curvature.

3. One of the properties of negative curvature is that you have a linear isoperimetric inequality, namely that any closed loop bounds a disk whose area is proportional to the length of the loop. In this problem, you'll show this for the hyperbolic plane $\mathbb{H}^2$.

To begin, let me clarify what it means for a loop $\gamma$ in $\mathbb{H}^2$ to bound a disk if $\gamma$ is not embedded. Let $S^1$ be the unit circle in $\mathbb{R}^2$, and $D$ the closed disk that it bounds. If $\gamma: S^1 \to \mathbb{H}^2$ is a loop, then a function $g: D \to \mathbb{H}^2$ such that $g = \gamma$ on $S^1$ is a disk that $\gamma$ bounds. The area of such a disk is:

$$\int_D |g^*(dA)|$$

where $dA$ is the area form on $\mathbb{H}^2$. The absolute value signs are in the integrand so this is the full area and not some kind of algebraic area.

Prove that there is a constant $C$ such that for every loop $\gamma$ in $\mathbb{H}^2$ bounds a disk whose area $\leq C \text{Length}(\gamma)$

4. In a continuation of the last problem, give an example of a complete Riemannian manifold which does not have a linear isoperimetric inequality.