1. Let \((M, g)\) be a Riemannian manifold, and \(R\) the curvature tensor. Show that for all tangent vectors \(a, b, c, d\) in \(T_pM\) we have \(R(a, b, c, d) = R(c, d, a, b)\). (This is the omitted part (iii) of Prop. 3.5 of GHL.)

2. Let \(S^n\) be the \(n\)-sphere with its standard round metric. Let \(\text{Isom}(S^n)\) be the group of Riemannian isometries of \(S^n\). Show that \(\text{Isom}(S^n) = O(n + 1)\).

3. A closed submanifold \(N\) of \((M, g)\) is called **totally geodesic** if every minimal geodesic with endpoints in \(N\) is contained in \(N\). That is, \(N\) is convex in the sense of the last HW.

   (a) Let \(N\) is totally geodesic submanifold of \(M\), and \(p\) a point in \(N\). Suppose that \(P\) is a plane in \(T_p(N)\). Show that \(K_N(P) = K_M(P)\), where the former is the sectional curvature measured in \(N\) (with the metric inherited from \(M\)), and the latter is the sectional curvature measured in \(M\).

   (b) Show that if \(N\) is not totally geodesic then the conclusion of part (a) need not hold.

   (c) Use part (a) to prove that sectional curvature of \(S^n\) is independent of \(n\), as claimed in class.

   (d) Suppose \(N\) is a closed submanifold of \(M\) which satisfies the conclusion of part (a) above. Does \(N\) have to be totally geodesic?

4. As explained in GHL §2.58, a connection on \(TM\) extends to a connection on the space of tensors on \(M\). Thus if \(R\) is the curvature tensor of type \((0,4)\), given a vector field \(X\) we can talk about its covariant derivative \(D_XR\) which is also a \((0,4)\) tensor. If we think of \(X\) as one of the inputs, then \(DR\) is a \((0,5)\) tensor.

Say that a Riemannian manifold \((M, g)\) is **algebraically locally symmetric** if \(DR = 0\) everywhere. A Riemannian manifold \((M, g)\) is **geometrically locally symmetric** if for each \(p\) in \(M\) there is a small ball \(B_p(\epsilon)\) so that map \(\exp(\nu) \mapsto \exp(-\nu)\) is an isometry on \(B_p(\epsilon)\). In later homeworks, you will show that these two conditions are equivalent; the class of Riemannian manifolds satisfying these conditions are called **locally symmetric**. This is one of the most important classes of Riemannian manifolds: It includes manifolds of constant curvature, \(\mathbb{C}P^n\), and compact Lie groups with biinvariant metrics. All of these examples just given are locally homogenous, and this is true of locally symmetric spaces in general.

   (a) Let \((M, g)\) be algebraically locally symmetric. Let \(c\) be a geodesic in \(M\). Let \(X, Y, Z\) be parallel vector fields along \(c\). Prove that \(R(X, Y)Z\) is also parallel.

   (b) Suppose \(M\) be a connected Riemannian 2-manifold which is algebraically locally symmetric. Prove that \(M\) has constant curvature. (The converse is true in any dimension.)