- Welcome! will discuss handout at end
- Note HW #1 on Handout.

**Topology:** Study of spaces up to homeomorphisms, as though made of rubber.

\[ \square \cong \triangle \cong \bigcirc \cong \]

**Topologist:** someone who can't tell a coffee cup from a doughnut.

\[ \text{\( \cong \) \bigcirc} \]

**Geometry:** Study of spaces w/ distance functions (metric spaces).

Ex: See above. \[ \text{Ex: } R^2 = d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \]

Excluding \[ d < \max(|x_1-x_2|, |y_1-y_2|) \]

[...and the cases, our focus will be on surfaces...

**Def:** A surface is a Hausdorff top space \( X \) s.t. every pt has an open nhbd homeomorphic to \( R^2 \).

Ex: \( R^2, \emptyset, \bigcirc, \bigcirc \bigcirc, P^2, K \)

Non Ex: \( R^3, \bigcirc \bigcirc \) < almost! a surface w/ \( \emptyset \).
Classification of Surfaces: Any compact surface $X$ is homeomorphic to exactly one of:

- $\mathbb{S}$, $\mathbb{O}$, $\mathbb{O}$, $\mathbb{O}$, $\mathbb{O}$, ....
- $\mathbb{P}^2$, $K = \mathbb{P}^2 \# \mathbb{P}^2$, $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$, ...

Geometry of surfaces: $X \subseteq \mathbb{R}^3$, smoothly embedded.

Extrinsic: how $X$ sits in $\mathbb{R}^3$

Intrinsic: distances measured within the surface

Different surfaces w/ same extrinsic geometry

Gaussian curvature:

- $K > 0$
- $K = 0$
- $K < 0$

Talk about: 1) Actually intrinsic vs. mean curvature in soap films

2) is a far of the ...
Connection between: Topology and geometry.

Euler-Chern: \( T \) a triangulated surface

\[
X(T) = \# \text{ of verts} - \# \text{ of edges} + \# \text{ of triangles}
\]

\[
X(\text{\begin{tabular}{c}
\end{tabular}}) = 4 - 6 + 4 = 2 \quad X(\text{\begin{tabular}{c}
\end{tabular}}) = 8 - 18 + 12 = 2!
\]

Thm: \( T_1 \) and \( T_2 \) are triang. of the same surface \( S \):

\[
X(T_1) = X(T_2)
\]

Def: \( X(X) = X(T) \).

\[
X(\text{\begin{tabular}{c}
\end{tabular}}) = 2 \quad X(\text{\begin{tabular}{c}
\end{tabular}}) = 0 \quad X(\text{\begin{tabular}{c}
\end{tabular}}) = -2
\]

Gauss-Bonnet: Suppose \( X \) is a exp. surface \( S \) in \( \mathbb{R}^3 \). Then

\[
(\text{Average of } K)(\text{Area of } K) = 2\pi X(S)
\]

Cor: Any \( \mathbb{R}^3 \) has foo curve somewhere.

Any \( \mathbb{R}^3 \) has many stars somewhere.

Discuss syllab., policies, etc.
Lecture 2:  • Note handout. Contains 1st HW due next Wed.

Last time:  Def: A surface in a top space s.t. each pt has an open nbhd $\cong \mathbb{R}^2$.

Classification Thm: A cpt surface is homeo to exactly one of

- $\varnothing$, $T = \varnothing$, $T \# T = \varnothing \circ \circ$, $\ldots$, $T \# \ldots \# T = \varnothing \circ \circ \ldots$
- $P$, $K = P \# P$, $P \# P \# P \ldots$

Today:
1) Connected sum (#)
2) Jordan curve thm and other top. issues.

**Connected sum:**

Def: A chart for a surface $S$ is an open set $U \cong \mathbb{R}^2$.

Def: A disc in $S$ is a cpt subset $D$ s.t. $\exists$ a chart $U \ni D$ where $U \cong \mathbb{R}^2$ takes $D \rightarrow \{x \in \mathbb{R}^2 | |x| \leq 1\}$

Def: $S_1, S_2$ surfaces. Then $S_1 \# S_2 = (S_1 \setminus D_1) \cup_f (S_2 \setminus D_2)$

where $D_i \subseteq S_i$ is a disc and $f : \partial D_i \rightarrow \partial D_2$ is a homeomorphism.

Need to check: 1) $S_1 \# S_2$ is a surface

2) $S_1 \# S_2$ is indep of the choices of $D_i, f$. 
For 1) need to check that pts along the join $c$ have $\text{nhbhd} \cong \mathbb{R}^2$

$$N(c) = U_1 \setminus D_1^0 \cup_f U_2 \setminus D_2^0$$
$$\cong S^1 \times [0, \infty) \cong S^1 \times [0, \infty)$$

\text{gluing $S^1 \times 0$ to $S^1 \times 1$ by some homeo.}

$$= S^1 \times (-\infty, \infty).$$

2) Breaks into 2 issues

a) coarse: find diff choices for $f$, reflection

$$\mathcal{Z}' = \text{Tr}_1(S^1) \longrightarrow \text{Tr}_1(S^1') = \mathcal{Z},$$

$$\text{id}_{\mathcal{Z}}(1) = 1$$

b) subtle: choices of $D_i$

$$r_{\mathcal{Z}}(1) = -1.$$

[a] is in some sense "accidental", a consequence of the dimension.

[b] I will deal with 2) in an odd way — it will prove the classification, thus avoiding this issue.

b) Thm $D_1 \cup D_2 \subseteq S$ then

$$\exists S \xrightarrow{f} S \text{ s.t. } f(D_1) = f(D_2).$$

Lemma 1: True if $D_2 \subseteq D_1$

Lemma 2: given $x, y \in S$, $\exists$

$$S \xrightarrow{f} S \text{ s.t. } f(x) = f(y).$$

Lemma 1 is on HW, but real work comes from:
Sörenflies Thm: Let $C$ be the image of $f : S^1 \rightarrow \mathbb{R}^2$. Then $h : \mathbb{R}^2 S \cong$ s.t. $h(C)$ is the unit circle $\{ x | |x| = 1 \}$.

Jordan Curve Thm: Any circle $C$ as above separates $\mathbb{R}^2$ into 2 regions. [Section 5.6 of Armstrong.]

Remarks: [clown pretty non-impressed by these things.]

1) Continuous functions are really necessary. [see handout on next page]

2) Analog is not true in dim 3! \exists f : S^2 \rightarrow \mathbb{R}^3 s.t. some comp of $\mathbb{R}^3 \setminus f(S^2)$ is not simply connected!

See other side of handout, discuss

Thm: Any cpt surface $S$ has a triangulation. [if time]

Comment on differing w/ text.
Lecture 3: Today and Wed: proving

Thm: Any connected surface is homeo to exactly one of
1) \( \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc \), ...
2) \( \bigcirc, \bigcirc, \bigcirc, \bigcirc \), ...

[Will use diff proof than other text, say why.]

Handle Decomposition:
1) Start w/ 0-handles = \( \mathbb{D}^2 \)
2) Glue 1-handles = \( \mathbb{I} \times \mathbb{I} \) to the boundary of the 0-handles along \( \partial \mathbb{I} \times \mathbb{I} \)
3) Glue 2-handles = \( \mathbb{D}^2 \) along whole of \( \partial \mathbb{D}^2 \)
   ... to every boundary comp of 0-1 handles

\( \Rightarrow \) gives a surface.

Ex:

Lemma: 5-ept conn surf. Then 5 has a handle dec w/ only one 0-handle and 2-handle. [and some unknown #] of 1-handles

Of: As S has a triangulation, it has a handle dec.
Suppose there are at least two 0-handles. Then \(B_1, B_2\) with two distinct 0-handles joined by a 1-handle. Now consider \(B_1 \cup B_2 \cup H\) as a 1-handle. [Check still leave a 1-handle decomposing.]

Similarly if have multiple 2-handles, find two adjacent across a 1-handle and then amyllogaminate.

To record such a hand decomposing, just need to draw 0 and 1 cells.

\[
\begin{align*}
\text{one 2-handle.} & \quad = \quad T^2 \\
0 & + 2\text{-handle.}
\end{align*}
\]

Two kinds of 1-handle

\[
\begin{align*}
\text{=} & \\
\text{=} & + 2\text{-handle.}
\end{align*}
\]
Def: To avoid sum issue

\[ T \# \cdots \# T := n \text{ times} \]

\[ P \# \cdots \# P = n \]

\[ n \quad + 2\text{-handle} \]

\[ n \quad + 2\text{-handle} \]

\[ \text{Pf of Claim: Consider a handle decomposition of } S \]
\[ \text{w/ one 0-handle, one 2-handle and } n-1 \text{-handles.} \]

**Orientable case:** no band is twisted.

**Claim:** \[ S \cong T \# \cdots \# T \]

\[ n/2 \]

\[ N = 0: \quad \includegraphics[width=1cm]{circle_0.png} + \includegraphics[width=1cm]{circle_2.png} = S^2 \]

\[ N = 1: \quad \includegraphics[width=1.5cm]{circle_1.png} \quad \text{not allowed as we would have to add two 2-handels to make a surface.} \]

\[ N = 2 \]

\[ \includegraphics[width=2cm]{circle_2.png} \]

\[ = T \checkmark \]
In general, suppose we have 1-handles \( h', h^2, \ldots, h^n \).

There must be some \( h^i \), say \( h^2 \), going from one leaf to the other.

**Key claim:** We assume \( h^i \) for \( i > 2 \) are all attached out here.

**Proof:** use handleslide.

Point: the boundary of \( B \) is just a circle.

Now repeat the argument on \( h^3, \ldots, h^n \), taking care to never leave them outside the region \( \Omega \).

This concludes the orientable case. [Query: Abel. 11]
Lecture 4: On side board: Class. the, lemma on existence of handle decomps, def of $T \# \ldots \# T$ as handles

Proof of lemma: $S$ a plnt ear surface. Choose a handle decomps w/ one $0$-handle and one $2$-handle.

Case 1: There are no twisted $1$-handles.

Claim: If there are $n$ $1$-handles then $S \cong T_{h_1, \ldots, h_n}^{n/2}$

$X_0 \cup X_1 = X_0 \cup h_1 \cup X_2 = X_1 \cup h_2 \cup \ldots \cup X_n = \cup h_n \cup D^2$

1) Doesn't matter what order we add the handles.

2) Can change handle structure of $X_K$ using handlesides

(Query: Not a handle)

[Doesn't change homeo type]

[Note: May have to move $h_{k+1}, \ldots, h_n$ slightly]

to get a handle decomps at the next stage.

3) If $\exists X_K$ in connected -- consists of just one circle -- then we can assume all remaining handles are glued to a segment of $\exists X_K$ we get to choose
4) If $\partial X_k$ has two components $C_1$ and $C_2$, then at least one $h_{k+1}, \ldots, h_n$ has one end on $C_1$ and the other on $C_2$.

**Proof of claim:**

\[ \begin{array}{c}
X_1 \\
\xrightarrow{h_1} z, 4 \\
\xrightarrow{3} \quad 3 \\
\text{all } h_{123} \text{ attached here.}
\end{array} \]

Repeat until get \[ T \# T \# \cdots \# T. \]

**Claim:** Suppose there are $n$ 1-handles at least one of which is twisted. Then $S \cong P \# \cdots \# P$. 

**Pf:** HW.
To complete the proof need to show all these are distinct.

\[ \Pi_1(\#T) = \langle a_1, b_1, \ldots, a_n b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_n b_n a_n^{-1} b_n^{-1} = 1 \rangle. \]

\[ \Pi_1(\#P) = \langle a_1, \ldots, a_n \mid a_1^2 a_2^2 \ldots a_n^2 = 1 \rangle. \]

**Problem:** These are different, but how do we prove it?

**Def:** \( G \) is a gp. Then \( G' = \text{gp gen by } \langle g_1, g_2 \rangle \) for all \( g_1, g_2 \in G \).

Set \( G^{ab} = G/G' \), the abelianization of \( G \). [Why is this a normal subgroup?] [why abelian?]

\[ \Pi_1(\#T^n) = \mathbb{Z}^{2n} \quad \Pi_1(\#P) = \mathbb{Z}^{n-1} \oplus \mathbb{Z}/2. \]

To compute, add relations so that all the gens commute.

\[ Why \ is \ \# \ sum \ well-defined? \]

**cause issue:** two diff homeos of \( S^1 \) \( \xrightarrow{\text{id}} \) \( \xrightarrow{r = \text{reflection}} \)

**Point:** \( S \) a compact connected surface w/ one boundary circle, then \( \exists \) a homeo of \( S \) of which flips the 0-circle:

This works for any \( T \# \cdots \# T \).
A surface contains a Möbius band, e.g. it has a twisted handle, then just slide the boundary circle around the band to reverse the orientation.
**Lecture 5: Smooth surfaces in \( \mathbb{R}^3 \)**

**Def:** If \( U \subseteq \mathbb{R}^n \) then \( f: U \to \mathbb{R}^m \) is smooth if

- \( U \) is open
- all partial derivatives of \( f \) of all orders exist and are continuous.

The derivative of \( f \) at \( p \) is \( D_p f = \left( \frac{\partial f_1}{\partial x_1}(p), \ldots, \frac{\partial f_m}{\partial x_n}(p) \right) \).

where \( f(x) = (f_1(x), \ldots, f_m(x)) \).

Gives best linear approximation to \( f \):

\[
f(x) = f(x_0) + (D_{x_0} f)(x - x_0) + E(x - x_0)
\]

where \( E(x, x_0) \) s.t. \( |E(x, x_0)| \leq M |x - x_0|^2 \) for all \( |x - x_0| < \delta \).

**Def:** \( U, V \) open sets in \( \mathbb{R}^n \). A function \( f: U \to V \) is a diffeomorphism if it is bijective and \( f \) and \( f^{-1} \) are both smooth.

[A kind of homeomorphism]

**Note:** If \( f \) is a diffeo, then \( \forall p \in U, D_p f \) is non-singular.

**Pf:** \( D_{f(p)} f^{-1} \circ D_p f = D_p (f^{-1} \circ f) = D_p (\text{Id}) = I \).

**Ex:** A non-diffeo: \( f: \mathbb{R}^5 \to \mathbb{R} \) \( f(x) = x^3 \) a smooth function

\( f^{-1}(x) = x^{1/3} \) not diff at 0.

**Inverse Function Theorem:** \( f: (U \subseteq \mathbb{R}^n) \to \mathbb{R} \) smooth.

- \( U \subseteq \mathbb{R}^n \) is such that \( D_p f \) is invertible.
- \( f \) a open nbhd \( W \) of \( U \) such that \( f(W) \) is open and \( f: W \to f(W) \) is a diffeomorphism.
Def: \( U \subseteq \mathbb{R}^2, f: U \to S \subseteq \mathbb{R}^3 \) a smooth map.

Then \( f \) is a coordinate patch if

1) \( f \) is a homeo from \( U \) to \( f(U) \), and \( f(U) \) is open in \( S \).

2) \( Df \) is 1-1 for each \( p \in U \)

Let us define a tangent plane.

Def: \( S \subseteq \mathbb{R}^3 \) is a smooth surface if for each \( p \in S \), there is a coordinate patch \( f: U \to S \) with \( p \in f(U) \).

Note: such an \( S \) is also a topological surface in the old sense [not that \( U \neq \mathbb{R}^2 \)]

[Def differ from text, doesn't require it is a homeo.]

**Ex:** \( U \subseteq \mathbb{R}^2 \), \( h: U \to \mathbb{R} \) smooth fn

\[
\begin{align*}
S &= \{(x,y,h(x,y)) \mid (x,y) \in U\} \\
\Phi(x,y) &= (x,y,h(x,y)) \quad \text{a coor. patch}.
\end{align*}
\]

\[
D\Phi = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & h_x \\
0 & 0 & h_y
\end{pmatrix}
\]

**Ex:**

\[
h = \sqrt{1 - (x^2 + y^2)}
\]
Lecture 6: Lease time: Def of smooth surface
   Today: Change of coor. lemma
   Smooth maps between surfaces, differ

- Change of coor. Lemma: \( S \subseteq \mathbb{R}^3 \) a smooth surface.
- Let \( f: U \to S \), \( g: V \to S \) coordinate charts.
- Let \( W = f(U) \cap f(V) \). Then \( f^{-1} g \circ g^{-1} f : W \to f(W) \) is smooth.

**Pf:** First, suppose \( f \) is a
   - Menge chart, as in the HW.
   - Say \( f(x,y) = (x,y,h(x,y)) \),
   - and let \( \pi: \mathbb{R}^3 \to \mathbb{R}^2 \) be proj onto the \( xy \) plane. Then
     \[ f^{-1} g = \pi \circ g \circ f \]
     which is the composite of two smooth functions,
     hence smooth. In general, we need

**Sublemma:** \( f \) a coor. patch, \( p \) a point in \( f(U) \). Then \( f^{-1} \circ p \) open \( U \to f(U) \circ p \) and a proj for \( \pi \) s.t. \( \pi \circ f \) is a diff eq on \( U \).

Thus \( (\pi \circ f)^{-1} \pi : \text{open in } \mathbb{R}^3 \to U \circ f \) is a smooth fn and restrict to \( f(U) \), it is just \( f^{-1} \). This case is

then just the same as before.
Def: \( S_1, S_2 \) are smooth surfaces. Then \( \varphi: S_1 \to S_2 \) is smooth if \( \forall \) cor patches \( f: U \to S_1, g: V \to S_2 \) we have \( g^{-1} \circ \varphi \circ f: (\varphi \circ f)^{-1}(g(V)) \to V \) is smooth.

[By change of cor lemma, onto need to check cond for some collection of charts (which cover \( S_1 \) and \( S_2 \)].

Def: A map \( \varphi: S_1 \to S_2 \) is a diffeomorphism if it is a bijection and \( \varphi \) and \( \varphi^{-1} \) are smooth.

[smooth analog of homic]:

Q: Is every topological surface a smooth surface in \( \mathbb{R}^3 \)?

A: [Query] No. After all some don’t even embed topologically in \( \mathbb{R}^3 \), e.g. \( P \) or \( K \). [But this is a silly reason could just use \( \mathbb{R}^4 \)].

Abstract Smooth Surface: is a topological surface \( S \) with a collection of homeos \( f_\alpha: (U_\alpha^{open} \subseteq \mathbb{R}^2) \to (\text{open subset of } S) \) s.t. \( f_\beta^{-1} \circ f_\alpha: f_\alpha^{-1}(f_\beta(U_\beta)) \to f_\beta^{-1}(f_\alpha(U_\alpha)) \) is smooth.

[I.e. defining exactly so the change of cor lemma holds].
Thm: $S$ a topological surface. Then there exist a cell $(f_x, U_x)$ making it into a smooth surface. Any two such smoothings are diffeomorphic. [Thus class of surfaces doesn't change.]

[Proof: For existence, use a triangulation and do the gluings in a controlled way.]

Note: False in higher dimensions: $S^7 = \{ x \in \mathbb{R}^8 | |x| = 1 \}$ [Query]

has 28 non-diffeomorphic smoothings.

Suppose $c : (-\varepsilon, \varepsilon) \to S^7$ is a smooth curve with $c(0) = p$. Then $c'(0) \in \mathbb{R}^3$ is called a tangent vector to $S$ at $p$. The collection of all such tangent vectors is the tangent space $T_p S$.

Lemma: Suppose $f : U \to S$ is a con patch with $f(x_0) = 0$.

Then $T_p M = \text{image} \left( D_{x_0} f \right)$ is always 2 dim'l.

[In particular, $T_p M$ is a 2 dimensional linear subspace of $\mathbb{R}^3$]
Suppose $V = D_{x_0}f(w)$. Let $\beta(t) = x_0 + tw$

Then $\alpha' = D_{x_0}f(\beta'(0)) = V$

by chain rule.

Let $\alpha$ be a curve defining a tangent vector $V$.

Shrink $\varepsilon$ so that image lies in $f(U)$.

Then $\beta = f^{-1} \alpha$ is smooth and

$\alpha = f \circ \beta$ so

$V = \alpha'(0) = D_{x_0}f(\beta'(0))$ as desired.
Lecture 7: Today: Geometry of curves.

[Quick comment:]

Smooth curve: smooth fn \( c: (a, b) \rightarrow \mathbb{R}^3 \).

Regular curve: \( c'(t) \neq 0 \) for all \( t \in (a, b) \).

[regular curve is analogous to our smooth surface; only need one chart as the topology of a manifold is trivial. Need to avoid.]

Oddly smooth curve: \( c: (-1, 1) \rightarrow \mathbb{R}^2 \) whose image is

\[
h: \mathbb{R} \rightarrow \mathbb{R} \quad \text{smooth w/ graph}
\]

\[
h(x) = \begin{cases} 
0 & x < 0 \\
\frac{f(x)}{f(x) + f(1-x)} & x \in [0, 1] \\
1 & x > 1
\end{cases}
\]

\( f(x) = e^{-\frac{1}{x}} \)

Take \( c(t) = (h(t), h(t+1)) \) to trace out.

W.B.: \( \exists \) a smooth map \( \mathbb{S} \rightarrow \mathbb{R}^3 \) w/ image \( \mathbb{S} \)
**Def:** A smooth curve \( C_2 : (a_2, b_2) \to \mathbb{R}^3 \) is a reparameterization of \( C_1 : (a_1, b_1) \to \mathbb{R}^3 \) if \( \exists \) a diffeomorphism \( h : (a_2, b_2) \to (a_1, b_1) \) s.t. \( C_2 = C_1 \circ h \)

\[
\text{Fix } C : (a, b) \to \mathbb{R}^3 \text{ a regular curve}
\]

\[
T(t) = \frac{C'(t)}{|C'(t)|} \quad \text{unit tangent vector}
\]

**Note:** Can reparameterize so that \( C \) moves at unit speed and \( C'(t) = T(t) \).

\[
d(t_0, t_1) \text{ along } C \text{ is } \int_{t_0}^{t_1} |C'(t)| \, dt = d(t)
\]

\( d \) is smooth, strictly increasing, so gives a diffeomorphism \( d : (a, b) \to (a', b') \)

Then \( C_{d'} = C_0 \circ d^{-1} \) is a unit speed parametrization.

For now, let's focus on a unit speed curve.
Curvature: quantitative measure of how bent the curve is at each point.

\[ K = \frac{1}{r} \]

**Def:** A unit-speed curve, then \( K(t) = |T'(t)| \)
\[ = |C''(t)| \]

**Ex:** Line has \( K = 0 \)

**Ex:** \( \gamma = (r \cos \frac{\theta}{r}, r \sin \frac{\theta}{r}) \)

\[ C'(t) = (-\sin \frac{\theta}{r}, \cos \frac{\theta}{r}) \] unit speed.

\[ C''(t) = (-\frac{1}{r} \cos \frac{\theta}{r}, -\frac{1}{r} \sin \frac{\theta}{r}) = -\frac{1}{r^2} C(t) \]

**Geometrically:**
1) \( \frac{1}{K} = \) turning radius

2) \( K(t) = \frac{1}{r} \) where \( r \) is the radius of the unique round circle at \( C(t) \)
whose first two curvatures match w/ \( C', C'' \).

3) \( K \) measures change in length as we push in the direction of the normal
Detail: [Acceleration is proportional to $T$ as speed is not changing.]

$$\mathbf{a} = \frac{d}{dt} \langle T, T \rangle = \langle T', T \rangle + \langle T, T' \rangle = 2 \langle T, T' \rangle$$

Set $N(t) = \frac{T'}{|T'|}$

$$\mathbf{C}'' = \mathbf{K} N$$

Consider the family of curves $C_s(t) = C(t) + sN(t)$

$$C: (a,b) \times \mathbb{R} \to \mathbb{R}^3$$

A smooth function.

$$L(s) = \int_a^b \left| \frac{dC_s(t)}{dt} \right| dt$$

Length of $C_s$

$$\left. \frac{dL}{ds} \right|_{t=0} = -\int_a^b \mathbf{K} \, dt$$

$K = \frac{1}{s}$

$$\mathbf{K} = \frac{1-K_s}{l}$$
Lecture 8: Last time: \( c \)

\( T = c'(t) \)

unit speed

\[ K(t) = |T'(t)| = |c''(t)| \]

\[ N = \frac{c''(t)}{|c''(t)|} \]

\[ KN = T' \]

Constant by type

\( (s, u, v, w, t) \)

unmeasured:
how \( c \) twists
wrt the osculating plane.

Set \( B(t) = T(t) \times N(t) \)

[Binormal]

Consider \( B'(t) \), and note:\n\[ \langle B', B \rangle = 0 \] [same reason as last time]

Also \[ \langle B', T \rangle = 0 \] because

\[ 0 = \frac{d}{dt} \langle B, T \rangle = \langle B', T \rangle + \langle B, T' \rangle = \langle B', T \rangle + \langle B, KN \rangle \]

Hence \( B' \) is a scalar mult of \( N \), say

\[ B'(t) = \kappa(t) N(t) \]

torsion of \( c \) at \( t \).

\[ \kappa(t) \]

Note: \( \kappa(t) \)
$C$ is strongly regular if $K(t) \neq 0$ for all $t$.

**Thm:** For a strongly regular unit speed curve,

\[
T' = KN
\]

\[
N' = -K T + \tau B \quad \text{for each } t.
\]

\[
B' = -\tau N
\]

**Pf:** \( \langle N', N \rangle = 0 \) for the usual reason.

\[
0 = \frac{d}{dt} \langle N, T \rangle = \langle N', T \rangle + \langle N, T' \rangle \implies \langle N', T \rangle = -K.
\]

\[
0 = \frac{d}{dt} \langle N, B \rangle = \langle N', B \rangle + \langle N, B' \rangle \implies \langle N', B \rangle = -\tau.
\]

**Q:** To what extent does $K, \tau$ determine $c$?

**Ex:** $c$ lies in a plane $\iff T = 0$ for all $t$. $\iff B$ is const

\[
\frac{d}{dt} \langle c(t) - c(t_0), B(t) \rangle = \langle c'(t), B(t) \rangle + \langle c(t) - c(t_0), B'(t) \rangle = 0.
\]

$\implies \langle c(t) - c(t_0), B(t) \rangle = 0$ for all $t$, so lies in a plane.
Fundamental Theorem of Curves:

Let \( K, T : (a,b) \to \mathbb{R} \) be smooth funs, \( w^1 K > 0 \).

Then \( \exists \) a strongly regular curve \( c : (a,b) \to \mathbb{R}^3 \)

\( w^1 \) curvature and torsion funs equal to \( K \) and \( T \).

This curve is unique up to translation and rotation.

Idea: (*) are a set of differential equations

for \( T, N, B \). For general reasons, they have a solution, say without loss of generality.

Thus set \( C = \int_t^b T(t) \, dt \)

\( T(t_0) = (1,0,0) \)

\( B(t_0) = (0,1,0) \)

\( N(t_0) = (0,0,1) \)

Check that \( T_c, N_c, B_c = T, N, B \).

and \( T, N, B \) are orthonormal.

What about non-unit speed curves?

\[
T = \frac{c'}{|c'|} \quad B = \frac{c' \times c''}{|c' \times c''|} \quad N = B \times T
\]

\[
K = \frac{|c' \times c''|}{|c'|^3}
\]

\[
\tau = \frac{\langle c' \times c'', c''' \rangle}{|c' \times c''|^2}
\]

The reason that you don't solve for \( N \) in terms of \( c'' \)

is that if \( c = C u^q \) \( \quad \text{unit speed} \quad \)

\( c'' = (C u'(g(t)) g(t))' = C u'(g(t))(g'(t))^2 + C u'(g(t)) g''(t) \)
Lecture 9: Today: Length and area of surfaces.

Length:
\[ c: (a, b) \rightarrow S \text{ curve in surface } S. \]

Length of \( c = \int_a^b |c'(t)| \, dt \quad c'(t) \in T_{c(t)} S \]

[Can also talk about angles of vectors in \( T_{c(t)} S \); both are encoded in this]

[Intrinsic geometry all comes from this information.]

Def: \( S \subseteq \mathbb{R}^3 \) a smooth surface. The first fundamental form of \( S \) at \( p \) is the fn \( I_p: T_p S \times T_p S \rightarrow \mathbb{R} \) defined by
\[ I_p(v, w) = \langle v, w \rangle \]

[In general, the first fund. form is the collection of all such.]

\( I_p \) is a symmetric bilinear form. [Query]

In a coordinate patch:
\[ e_1 = (1, 0) \]
\[ e_2 = (0, 1) \]
\[ \overline{e}_1 = f_x \]
\[ \overline{e}_2 = f_y \]

\[ g_{ij}(u) = I_p(D_u f(e_i), D_u f(e_j)) \]
\[ g_{ii} = \text{length} \left(D_u f(e_i)\right)^2 \]

\[ V = (v_1, v_2) = v_1 e_1 + v_2 e_2 \]
\[ W = (w_1, w_2) = w_1 e_1 + w_2 e_2 \]
\[ I_p(v_1 \overline{e}_1 + v_2 \overline{e}_2, w_1 \overline{e}_1 + w_2 \overline{e}_2) \]

\[ I_p(D_u f(v), D_u f(w)) = v_1 w_1 g_{11} + v_1 w_2 g_{12} + v_2 w_1 g_{21} + v_2 w_2 g_{22} \]
\[ D_u f(v_1 e_1 + v_2 e_2) = v_1 \overline{e}_1 + v_2 \overline{e}_2 \]
\[ = 1 \sqrt{\left( g_{11} g_{22} \right) \omega^T} \quad \text{Note: } g_{21} = g_{12} \text{ as } \Im \text{ is symmetric.} \]

**Ex:** Suppose \( c \) is a curve in \( S \), with image in a patch \( f: U \rightarrow S \).

\[
\text{Length}(foc) = \int_a^b ||(foc)'(t)|| \, dt = \int_a^b \sqrt{\text{I}_p((foc)'(t), (foc)'(t))} \, dt
\]

\[
= \int_a^b \sqrt{c'(t)^T \mathbf{G} c'(t)} \, dt.
\]

**In**

Area: \( A \subseteq f(U) \subseteq S \), the area of \( A \) is

\[
\int_{f^{-1}(A)} |\mathbf{e}_1 x \mathbf{e}_2| \, dx_1 \, dx_2
\]

\[
= \int_{f^{-1}(A)} \sqrt{\det \mathbf{G}} \, dx_1 \, dx_2 \quad \text{(should it exist)}
\]

\[
|\mathbf{e}_1^2| |\mathbf{e}_2^2| \cos^2 \theta
\]

\[\text{as } \det \mathbf{G} = g_{11} g_{22} - g_{12}^2 = |\mathbf{e}_1|^2 |\mathbf{e}_2|^2 - \langle \mathbf{e}_1, \mathbf{e}_2 \rangle^2
\]

\[= |\mathbf{e}_1|^2 |\mathbf{e}_2|^2 (1 - \cos^2 \theta) = |\mathbf{e}_1 \times \mathbf{e}_2|.
\]
Lemma: Area does not depend on which chart you use.

Note: Layer sets can be broken into pieces lying in charts in order to compute the area.

Normal vectors:
\[ \mathbf{n} = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|} \]
a normal vector to \( S \) at \( p \) is one \( \perp \) to \( T_p S \). [Usually look at unit unit normal vector may not be able to do so globally.]

Gauss Map: [tool to define curvature.]

\[ \hat{\mathbf{n}} : S \to S^2 = \{ x \in \mathbb{R}^3 \mid |x| = 1 \} \]
consistent choice of unit normal at each \( p \).

Then set \( \hat{\mathbf{n}} : S \to S^2 \) where \( p \) maps to unit normal \( \hat{\mathbf{n}}(p) \) at \( p \).
What does it mean for \( \phi: S \to \mathbb{R} \) to be smooth? Or \( \phi: S_1 \to S_2 \)?

**Def:** \( \phi: S \to \mathbb{R} \) is smooth if for every coordinate patch \( f: U \to S \) we have \( \phi \circ f: U \to \mathbb{R} \) is smooth.

**Ex:** If \( W \) is an open set \( \subseteq S \), \( \phi: W \to \mathbb{R} \) is smooth, then \( \phi: S \to \mathbb{R} \) is also smooth.

Note: It suffices to check this condition for a collection of coordinate charts that cover \( S \) because of

**Change of Coordinates Lemma**: \( S \subseteq \mathbb{R}^3 \) a surface.

Let \( f: U \to S \), \( g: V \to S \) coordinate charts.

Let \( W = f(U) \cap f(V) \). Then

\[
 f^{-1} \circ g: g^{-1}(W) \to f^{-1}(W) \text{ is smooth.}
\]

Is \( \phi \circ g \) differentiable at \( y \in f^{-1}(W) \)? Yes, as

\[
 \phi \circ g = (\phi \circ f)(f^{-1} \circ g)
\]
Lemma: Let $S$ be a smooth surface in $\mathbb{R}^3$, given $p \in S$, we can permute the coordinates so that there is a coor patch $f: U \to S$ w/ $f(u) \in p$ of the form $f(x, y) = (x, y, h(x))$.

Proof: Take some coor patch containing $p$. Let $T = \text{image}(Dg^{-1}(p) \delta)$.

By permuting the vars., can assume the proj. $p: \mathbb{R}^3 \to \mathbb{R}^2$ is $(x, y, z) \mapsto (x, y)$ surjective when restricted to $T$.

Then by the inverse function theorem, $p \circ g$ is a diffeo when restricted to some nbhd $W$ of $g^{-1}(p)$. Take $f = g \circ (p \circ g)^{-1}$.

Why does it imply the change of coor lemma?
Last time: \( S \subseteq \mathbb{R}^3 \) smooth surface, with consistent unit normal

**Gauss map:** \( \hat{n} : S \rightarrow S^2 = \{ x \in \mathbb{R}^3 \mid |x| = 1 \} \)

- \( p \rightarrow \) unit normal at \( p \).

[Examples from last time, plane, cylinder, sphere,...]

\( \hat{n} \rightarrow \) tangent spaces agree along

"Turned over like a pancake."

\( n(A_1) \rightarrow n(A_2) \)
Gauss curvature: \[ K(p) = \lim_{A \to p^1} \frac{\text{Area}_0(\hat{n}(A))}{\text{Area}(A)} \]

Where \( A = f(U) \) a coordinate patch,

\[
\text{Area}(A) = \int_{f^{-1}(A)} \langle f_1 \times f_2, n \rangle \, dx \, dy \\
\text{Area}_0(A) = \int_{f^{-1}(A)} \langle n_1 \times n_2, n \rangle \, dx \, dy
\]

\[
f_1 = \frac{\partial f}{\partial x} = e_1 \\
f_2 = \frac{\partial f}{\partial y} = e_2 \\
n_1 = \frac{\partial n}{\partial x} \\
n_2 = \frac{\partial n}{\partial y}
\]

\[
D_p \hat{n}(f;.) = n_i
\]

\[ K = 0 \quad \implies \quad \text{Dilating the surface } S \to rS \text{ changes } K(rS, r_p) = \frac{1}{r^2} K(S, p). \]
Problem: Is this well defined? How do we compute?? [Note in nature]

Alternate approach:

Weingarten map:

\[ D_p \hat{\n} : T_p S \rightarrow T_{\hat{n}(p)} S^2 \]

Note: These are the same plane! If we identify them, get
\[ L : T_p S \rightarrow T_p S \] a linear map.

Ex:

\[ L = 0 \]  

Note: extrinsic nature

\[ L = \frac{1}{R} I \]
L: $V \rightarrow W$ doesn't have many variants.
L: $V \rightarrow V$ has a lot V and $tr L$

**Def:** Gaussian curvature at $p$, $K(p) = \det L$
Mean curvature at $p$, $H(p) = \frac{1}{2} tr L$

[explain why this makes sense rel our earlier discussion]

**Ex:**

$L = 0$, $K = 0$, $H = 0$

$L = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $K = 0$, $H = \frac{1}{2}$

$K = \frac{1}{R^2}$, $H = \frac{2}{R}$

What is mean curve good for?

**Def:** A surface is minimal if $H(p) = 0$ $\forall p$.

**Ex:** Soap bubble surface.

Existence: Plateau's problem / Douglas, Rado 1930
Lecture II: Last time: \( \hat{N}: S \to S^2 \) Gauss map

Midterm handed out on Wed.
Open notes, book.

Weingarten map.
Gaussian curvature: \( K(p) = \det L \) [def. def. of area]
Mean curvature: \( H(p) = \frac{1}{2} \tr L \)

Today: more about \( L \), and how \( K, H \) relate to curvature of curves.

Lemma: \( L \) is self-adjoint, i.e. \( \langle Lv, w \rangle = \langle v, Lw \rangle \)
Equivalent, the matrix of \( L \) w.r.t. an orthonormal basis is symmetric.
\[
\begin{bmatrix}
e_1, e_2 \text{ new Lij} = \langle Le_j, e_i \rangle \end{bmatrix}
\]

Proof: Let \( f: U \to S \) be a chart with \( f(0) = p \).
May assume \( v \) and \( w \) are linearly indep and
\[
v = e_1 = \frac{\partial f}{\partial x} \hspace{1cm} w = e_2
\]
Set \( N = \hat{N} \circ f: U \to S^2 \)

\[
L(v) = D_p \hat{N}(v) = D_0 N(e_1) = \frac{\partial N}{\partial x}
\]
\[
L(w) = \frac{\partial N}{\partial y}
\]

Note:
\[
\left\langle \frac{\partial f}{\partial x}, N \right\rangle \bigg|_{0} = 0 \quad \Rightarrow \quad \left\langle \frac{\partial f}{\partial y \partial x}, N \right\rangle + \left\langle \frac{\partial f}{\partial x}, \frac{\partial N}{\partial y} \right\rangle = 0
\]
\[
\left\langle \frac{\partial f}{\partial y}, N \right\rangle = 0 \quad \Rightarrow \quad \left\langle \frac{\partial f}{\partial y \partial x}, N \right\rangle + \left\langle \frac{\partial f}{\partial x}, \frac{\partial N}{\partial x} \right\rangle = 0
\]
\[
\Rightarrow \text{ at } p \quad \langle v, L(w) \rangle = \langle w, L(v) \rangle.
\]
Def: The 2nd fundamental form at $p$ is defined by

$$\Pi_p : T_p S \rightarrow T_p S$$

$$(v, w) \mapsto \langle L(v), w \rangle$$

$$\Pi_p (v, w, +w_e) = \langle L(w), v \rangle = \langle w, L(v) \rangle = \langle L(v), w \rangle$$

Note: $\Pi_p$ is bilinear and symmetric.

What does $\Pi_p$ measure??

Normal curvature of $c$ at $p$:

$$K_n = -K \langle N_c, n \rangle = -K \cos \theta.$$  

Curvature of $c$ at $p$  \[\text{[Measure arclength curvature.]}\]

Thm: $\Pi_p (c', c') = K_n$.  \[\text{[Not: only depends on $c'$!]}\]

Proof: Assume $c$ is unit speed. Let $n(t)$ be the normal to $S$ at $c(t)$

As $\langle n(t), c'(t) \rangle = 0$ we have

$$\langle n'(t), c'(t) \rangle = -\langle n(t), c''(t) \rangle = -\langle n(t), K(t) N_c(t) \rangle = K_n$$

$$\langle D_p \, \hat{n} (c'(t)), c'(t) \rangle = \Pi_p (c', c')$$

$$\Pi_p (e_1, e_1) = -K(\alpha)$$

$$\Pi_p (e_2, e_2) = -K(\beta)$$

$$L(e_1) = -K(\alpha) e_1$$

$$L(e_2) = K(\beta) e_2$$
Hence: 
\[ K(p) = -K(\alpha) K(\beta) \]
\[ H(p) = \frac{1}{2} (-K(\alpha) + K(\beta)) \]

In general as \( L \) is symmetric, \( \exists \) an orthonormal basis \( e_1, e_2 \) of \( T_p S \) so that \( L \) is diagonal \( L = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \) with \( K_1 \geq K_2 \), principal curvatures.

For any unit vector \( v = \sin \theta e_1 + \cos \theta e_2 \), we have
\[ \Pi_p(v, v) = \langle L(v), v \rangle = \langle K_1 \sin \theta e_1 + K_2 \cos \theta e_2, v \rangle \]
\[ = K_1 \sin^2 \theta + K_2 \cos^2 \theta \]

**Geometrically:**
\[ K_1 = \max \text{ normal curvature of all curves } c \subset S \]
passing through \( p \).
\[ K_2 = \min \]

Because there are two polarities for \( p \),
\[ \text{Note: } L \text{ is only defined up to sign}, \text{ so is } H, K_1, K_2 \]
\[ K \text{ is well defined in general}. \]

\[ K_1, K_2 \]
Lecture 12: Intrinsic vs. Extrinsic.

Def: \( \varphi: S_1 \to S_2 \) be a smooth map of surfaces in \( \mathbb{R}^3 \).

Then \( \varphi \) is a **local isometry** if for all \( p \) and \( v, w \in T_p S_1 \) we have

\[
I_{S_1, p} (v, w) = I_{S_2, \varphi(p)} \left( D_p \varphi(v), D_p \varphi(w) \right).
\]

\( \varphi \) is an isometry if it is also a diffeomorphism.

Def: A property is **intrinsic** if it is invariant under isometries.

Intrinsic:
- Length of a curve
  
  \[
  \text{len}(c) = \text{len}(\varphi \circ c)
  \]

  \[
  \int_a^b \sqrt{I_{c(t)}(c'(t), c'(t))} \, dt = \int_a^b \sqrt{I_{c(t)}(\varphi(c'(t)), \varphi(c'(t)))} \, dt
  \]

- Area [come back to this]

Extrinsic:
- Dist between points in \( \mathbb{R}^3 \)
- Mean curvature

Notes:
- Local isometries are local diffeomorphisms [Query?]
- \( \varphi \) is a local isom if \( \forall \) charts \( U \to S_1 \) we have

\[
\mathbf{g}_{ij} = \mathbf{g}_{ij}' / \text{metric coeff for } U \to S_1
\]

metric coeff for \( I_{S_1} \) [relate back to area.]
**Theorem Egregium** ("remarkable theorem")

\[ \varphi : S_1 \to S_2 \text{ is a local isometry. Then } \forall p \in S_1, \text{ we have} \]

\[ K_{S_1}(p) = K_{S_2}(\varphi(p)) \]

[i.e. Gaussian curvature is **intrinsic**.]

[Explain why it is surprising.]

**Proof:** Express \( K \) in terms of \( g_{ij} \) and its derivatives.

\[ \vec{e}_1 = D f(e_1) = \frac{\partial f}{\partial x} = f_x, \quad \vec{e}_2 = f_y \]

\[ g_{ij}(u) = I_p \left( \vec{e}_i, \vec{e}_j \right) \quad g_{12} = g_{21} \]

\[ l_{ij}(u) = \Pi_p \left( \vec{e}_i, \vec{e}_j \right) \quad l_{12} = l_{21} \]

\[ (L_{ij}) = \text{matrix of Weingarten maps} \]

Focus on this as it is symmetric, whereas typically isn't.

\[ \Pi_p \left( \vec{v}, \vec{w} \right) = v^T (g_{ij}) w \]

\[ = \left( (L_{ij}) v \right)^T (g_{ij}) w \]

Taking transpose \( = v^T (L_{ij})^T g_{ij} w \implies (l_{ij}) = (L_{ij})^T g_{ij} \]

\[ \implies (l_{ij}) = (g_{ij}) (L_{ij}) \implies (L_{ij}) = (g_{ij})^{-1} (l_{ij}) \]

\[ \implies K = \text{det} (l_{ij}) = \frac{\text{det} (l_{ij})}{\text{det} (g_{ij})} = \frac{l_{11} l_{22} - l_{12}^2}{g_{11} g_{22} - g_{12}^2} \]

need can be expressed in terms of \( g_{ij} \)
Take the normal \( n = \frac{\bar{\mathbf{e}}_x \times \bar{\mathbf{e}}_z}{|\bar{\mathbf{e}}_x \times \bar{\mathbf{e}}_z|} : U \rightarrow S^2 \)

\[ f_{xx} = \Gamma_{11}^1 f_x + \Gamma_{12}^2 f_y - l_{11} n \]
\[ f_{xy} = \Gamma_{12}^1 f_x + \Gamma_{22}^2 f_y - l_{12} n \]
\[ f_{yy} = \Gamma_{22}^1 f_x + \Gamma_{22}^2 f_y - l_{22} n \]

\[ \frac{1}{2} (g_{11})_x = \langle f_{xx}, f_x \rangle = \Gamma_{11}^1 g_{11} + \Gamma_{12}^2 g_{12} \]
\[ (g_{11})_x - \frac{1}{2} \partial_y g_{12} = \langle f_{xx}, f_y \rangle = \Gamma_{11}^1 g_{12} + \Gamma_{12}^2 g_{22} \]

Same for rest \( \Rightarrow \Gamma_{jk}^i \) are determined by \( g_{ij} \).

A calculation then shows:

\[ l_{11} l_{22} - l_{12}^2 = \sum_{r=1}^{2} g_{1r} \left( \frac{\partial \Gamma_{22}^r}{\partial x} - \frac{\partial \Gamma_{21}^r}{\partial y} + \sum_{m=1}^{2} \left( \Gamma_{22}^m \Gamma_{m1}^k - \Gamma_{m1}^r \Gamma_{m2}^k \right) \right) \]

\[ \Rightarrow \text{Theorem: Egregium.} \]
Lecture B: Last time: Intrinsic v. Extrinsic
Today: Geodesics and distances.

Intrinsic Distance:

\[ d(p,q) = \inf \{ \text{len}(c) \mid \text{a path in } S \text{ joining } p \text{ to } q \} \]

Ex: This makes $S$ into a metric space.

Fact: Provided $S$ is closed in $\mathbb{R}^3$, $d(p,q) = \text{len}(c)$ for some particular $c$.

Variational Characterization:

\[ \text{[unit speed]} \quad c : [a,b] \rightarrow S \]
\[ C : (-\varepsilon, \varepsilon) \times [a,b] \rightarrow S \]
\[ \forall t \quad C(t,a) = p \quad C(t,b) = q \quad \varepsilon > 0 \]

\[ C_\alpha : [a,b] \rightarrow S \quad \alpha \in \mathbb{R} \]
\[ C_\alpha(t) = C(\alpha, t) \]
and $C_0 = c$

If $c$ is a geodesic, then \( \text{len}(c) \leq \text{len}(C_\alpha) \) for all $\alpha$. 

Talk about expectation.
Hence \[ \frac{d}{d\alpha} \left( \frac{\partial}{\partial \alpha} \left( \int_a^b \sqrt{\langle c'_a(t), c'_a(t) \rangle} \ dt \right) \right) \bigg|_{\alpha = 0} \]

\[ = \int_a^b \left( \frac{1}{2} \cdot 2 \left\langle \frac{\partial c}{\partial x}(x,t), \frac{\partial c}{\partial t}(x,t) \right\rangle \right) \bigg|_{\alpha = 0} \ dt \]

\[ = \int_a^b \left\langle V'(t), c'(t) \right\rangle \ dt \]

\[ V(t) = \frac{\partial c}{\partial \alpha}(0,t) \]

\[ = \left\langle V(b), c'(b) \right\rangle - \left\langle V(a), c'(a) \right\rangle - \int_0^b \left\langle V(t), c''(t) \right\rangle \ dt \]

\[ = -\int_0^b \left\langle V(t), c''(t) \right\rangle \ dt \]

So, \( c \) a geodesic \( \Rightarrow \) \( c''(t) \) is normal to \( S \) for all \( t \).

[Actually:]

Def: A geodesic in \( S \) is a curve \( c: [a,b] \to S \) such that \( c''(t) \) is normal to \( S \) for all \( t \).

Note: need not minimize length.

Note: is intrinsic, by our claim.
Do geodesics always exist?

Covariant differentiation:

\[ C: (a, b) \rightarrow S \text{ a curve} \]
\[ V: (a, b) \rightarrow \mathbb{R}^3 \text{ a vector field along } C, \text{ i.e. } V(t) \in T_{C(t)} S \]

\[ \frac{DV}{dt} (t) = \text{Projection onto } T_{C(t)} c' V' \]

Note: \( c \) is a geodesic iff \( \frac{Dc'}{dt} = 0 \) for all \( t \).

In local coors.

\[ V(t) = V_1(t) \frac{d}{dx_1} (c_1(t), c_2(t)) + \]
\[ V_2(t) \frac{d}{dx_2} (c_1(t), c_2(t)) \]

\[ V' = \sum_{i=1}^{2} \left( V_i f'_{x_i} + \sum_{j=1}^{2} \left( \frac{\partial}{\partial x_j} f(x_i x_j) \right) c_j' \right) \]

\[ f_{x_i x_j} = \Gamma_{ij}^1 f_{x_i} + \Gamma_{ij}^2 f_{x_j} - \delta_{ij} n \text{ throw away} \]

\[ \frac{DV}{dt} = \sum_{i=1}^{2} \left( V_i + \sum_{j,k=1}^{2} \Gamma_{jk}^i V_j c_k' \right) f_{x_i} \implies \frac{DV}{dt} \text{ is intrinsic.} \]
In local coordinates:

$$V(t) = v_1(t) f_{x_1}^x(c_1(t), c_2(t)) +$$

$$v_2(t) f_{x_s}^x(c_1(t), c_2(t)) +$$

$$V' = \sum_{i=1}^{2} \left( v'_i f_{x_i}^x + v_i f_{x_i x_1} c'_1 + v_i f_{x_i x_2} c'_2 \right)$$

$$\frac{DV}{dt} = \sum_{i=1}^{2} f_{x_i}^x \left( v'_i + \sum_{j,k=1}^{2} v_j f_{x_1}^x \Gamma^i_{jk} c'_k \right)$$
Proof: Assume coordinates \( f: U \to S \) with \( f(0) = p \).

Consider a curve in \( U \), \( (c_1(t), c_2(t)): (-\varepsilon, \varepsilon) \to U \)

\[
C'(t) = c_1'(t) f'_{x_1} + c_2'(t) f'_{x_2} + \text{?} \nu
\]

\[
\frac{Dc'}{dt} = 0 \iff C''_i = -\sum_{j,k=1}^2 \Gamma^i_{jk} C_j'(t) C_k'(t)
\]

Take \( d_i = C_i' \), get a 1st order system

\[
d_i = c_i', \quad d_i' = -\sum_{j,k=1}^2 \Gamma^i_{jk} d_j \, d_k
\]

By math 2a, there exist a unique solution to these equations with initial cond \( C_1(0) = c_2(0) = 0 \), \( d_1(0) = v_1, d_2(0) = v_2 \)

where \( v = v_1 f_{x_1} + v_2 f_{x_2} \).

Note: good may not exist for all time.

\[\square\]

Def: A symmetry of \( S \) is an isometry \( \Phi: S \to S \).

Cor: Suppose \( \Phi \) is a symmetry of \( S \) which fixes \( p \) in \( S \)

and \( v \in T_p S \). Then the geodesic through \( p \) w/ tangent vector \( v \) is pointwise fixed by \( \Phi \).

Proof: Let \( c \) be the specified good. Then \( \Phi \circ c \) is also a good...
and if \( c(0) = p \) then \( \varphi \circ c(0) = \varphi(p) = p \) 
\( c'(0) = v \) 
\[ (\varphi \circ c)'(0) = (D_p \varphi)(c'(0)) = v. \]

Thus: great circles are the geodesics (all) good.

- ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \)
- surface of revolution

**Exponential Map:**

\( \forall \nu \in T_p S. \) Set

\[ \rho_v = \sup \{ re \mathbb{R}^+ | \exists \) a good \( c : (-r, r) \to S \cup \{c(0) = p, c'(0) = v\} \}
\]
\[ [\rho_v = \infty] \]

**Note:**
- \( \rho_v > 0. \)
- \( \exists \) a good \( c : (-\rho_v, \rho_v) \to S \cup \{c'(0) = v\} \)
- \( s \in \mathbb{R} \setminus \mathbb{Q} \) then \( \rho_{sv} = \frac{\rho_v}{|s|}. \)

\[ E_p = \{ \nu \in T_p M | \rho_v > 1 \} \]

\[ \exp_p : E_p \to S \]

Distance traveled equals \(|v|\)

\( v \to c(t) \) where \( c \) is the geodesic such that 
\( c(0) = p \) and \( c'(0) = v. \)
Note: $\exp_p(sv): (-pv, pv) \rightarrow S$ is the geodesic through $p$ with tangent vector $v$.

Thm: For any $p$, $\exists$ an open set $0 \in U \subset E_p$ on which $\exp_p$ is smooth.

Pf: Math 2b.

Cor: $\exists U \ni 0$ in $T_p S$ such that $\exp|_U$ is a diffeomorphism.

Pl: $D_0(\exp_p) = \text{Id}$
Today: Gauss Lemma and nice local core.

\[ \exp_p : T_p S \to S \]
\[ \nu \mapsto C_{\nu}(1) \]
where \( C_\nu \) is the unique geod \( \nu \) with \( C_\nu(0) = p \) \( C_\nu'(0) = \nu \)

\[ S_r(p) = \exp_p( \text{circle of radius } r \text{ about } 0 ) = \{ x \in S | x \text{ can be joined to } p \text{ by a geod of length } r \} \]

\[ B_r(p) = \exp_p( \text{ball of radius } r \text{ about } 0 ) \]

\text{Note: for small } r, S_r(p) \text{ is a regular curve, analogous to a circle.}

\text{Thm: For small } r, S_r(p) \text{ is an intrinsic dist.}
\[ S_r(p) = \{ q \in S | d(p, q) = r \} \]
\[ [\text{Will show later.}] \]

\text{Thm: Length(} S_r(p) \text{) = } 2\pi r \left( 1 - \frac{K(p)}{6} r^2 + \text{higher order} \right) \]

\text{In particular}
\[ K(p) = \lim_{n \to 0} \frac{6}{r^2} \left( 1 - \frac{L_r}{2\pi r} \right) \implies K(p) \text{ is intrinsic.} \]
Cauchy Lemma: \( p \in S \) a smooth surface in \( \mathbb{R}^3 \).

Let \( c \) be a geod through \( p \). Then for all small \( r \),
\[ c \perp S_r(p). \] [Tells us a lot about \( D\exp_p \)]

**Plausibility argument:**

Choose an orthonormal basis \( e_1, e_2 \) of \( T_pS \)

**Good Polar Coordinates:** around \( p \).

\[
\begin{align*}
f: (0, r_0) \times (0, 2\pi) & \longrightarrow S \\
(r, \theta) & \longmapsto \exp_p \left( r \sin \theta e_1 + r \cos \theta e_2 \right)
\end{align*}
\]

For small \( r_0 \), this is a coordinate chart

[Note how \( \theta \) is constrained]
Key features:

$f((r, \theta_0)): (0, r_0) \to S$ is a good arc

$f((r_0, \theta)): (0, 2\pi) \to S$ is the circle $S_r(p)$.

Metric in local con:

$g_{ij}: U \to \mathbb{R}$

$g_{11} = 1$ everywhere as $|f_r(r_0, \theta_0)|^2 = |C_{\theta_0}'(r)|^2 = 1$

where $C_{\theta_0}'(r) = f(r, \theta_0) = \exp_p\left(r \left(\sin \theta_0 e_1 + \cos \theta_0 e_2\right)\right)$

$g_{12} = g_{21} = 0$ everywhere via Gauss' Lemma.

$= \langle f_r, f_\theta \rangle$

As really only one function $g_{22}$.

**Proof of Gauss' Lemma:** Will show $g_{12} = 0$.

$\frac{\partial}{\partial r} g_{12} = \frac{\partial}{\partial r} \langle f_r, f_\theta \rangle = \langle f_{rr}, f_\theta \rangle + \langle f_r, f_{r\theta} \rangle$

$= \langle f_r, f_{r\theta} \rangle = \frac{1}{2} \frac{\partial}{\partial \theta} \langle f_r, f_r \rangle = 0$

So for any fixed $\theta_0$, have $g_{12}(r, \theta_0) = 0$. Assume $\theta_0 = 0$. 

Now $A(r) = \frac{C}{2\pi r}$ and $A(0) = 0$. Thus $C = 0$ as desired.
Lecture 16: Last time: geod polar coords

\( e_1, e_2 \) orthonorm. basis for \( T_p S \)

\[ f: (0, R_0) \times (-\pi, \pi) \]

\[ \rightarrow \]

\[ \exp_p (r \cos \theta e_1 + r \sin \theta e_2) \]

\[ g_{11} = 1, \quad g_{12} = g_{21} = 0 \] everywhere.

---

Intrinsic distance:

\[ d(p, q) = \inf \{ \text{len}(c) \mid c \text{ a smooth curve} \} \text{ in } S \text{ joining } p \text{ to } q \]

\[ S_r(p) = \exp_p (\text{circle about } 0) = \{ \text{all pts joined to } p \} \text{ by geod of } \text{len} = r \]

Thm: \( p \in S \subseteq \mathbb{R}^3 \). Then \( \exists \varepsilon > 0 \) such that

\[ S_r(p) = \{ q \mid d(p, q) = r \} \text{ for } r < \varepsilon. \]

and \( \exists ! \) geod from \( p \) to each \( q \) of \( S_r(p) \) of \( \text{len} \ r. \)

Proof: Choose \( \varepsilon \) s.t. \( \exp_p: B_\varepsilon(o) \rightarrow S \) is a diffeo onto its image. [Note this immediately establishes]

Suppose \( c: [0, t_0] \) be a unit speed curve joining \( p \) to \( q \in S_{r_0}(p) \) where \( t_0 < r_0 < \varepsilon. \)

By changing \( \varepsilon \), can assume \( c([0, t_0]) \subseteq \exp_p(B_\varepsilon(o)) \)
Consider \( r(t) = \sqrt{c_1(t)^2 + c_2(t)^2} \)
and set \( (c_1', c_2') = c_r' + c_\theta' \)
in polar coordinates.

Thus \( |c'(t)| = |D\exp_p(c'_r) + D\exp_p(c'_\theta)| \)
\[ \geq |D\exp_p(c'_r)| = |c'_r| \]
\[ = |r'(t)| \]

Hence \( \text{len}(c) = \int_0^t |c'(t)| \, dt \)
\[ \geq \int_0^t |r'(t)| \, dt \geq \int_0^t r'(t) \, dt \]
\[ = r(t_0) - r(0) = r_0 \]

This contradicts that \( \text{len}(c) = t_0 < r_0 \).

Thm. \( p \in S \subseteq \mathbb{R}^3 \) Then \( \exists \varepsilon > 0 \) and \( U \) open nbhd of \( p \) s.t.

\( A \, q_1 \in U \), \( \exp_{q_1} |_{B_\varepsilon(0)} \) is a diffeo onto its image. \( \varepsilon \)

\( q_2 \in S \) and \( d(q_1, q_2) < \varepsilon \) then \( \exists \) a unique geod.
from \( q_1 \) to \( q_2 \) whose length is \( d(q_1, q_2) \).
Pf: First sentence follows from O.D.E. theory. See Prop 8.2.3. Rest follows from preceding.

Thm: Suppose $S \subseteq \mathbb{R}^3$ is a smooth surface which is closed in $\mathbb{R}^3$. Then $A_p$, $\exp_p$ is defined on all of $T_p S$.

Pf: Suppose not, and there is $p$ and a unit vector $v \in T_p S$.

$$\rho = \sup \{ t_0 \mid \text{exists } c : (s, t_0) \rightarrow S \text{ with } c(0) = p, c'(0) = v \}$$

is $< \infty$. Let $c : (s, \rho) \rightarrow S$ be the maximal geod.

with $c(0) = v$, $c(s) = p$. Note that $\text{image}(c) \subseteq B_{\rho}^{\mathbb{R}^3}(p)$.

Let $K = S \cap B_{\rho}(p)$, with $K$ compact.

By Thm, $\exists \varepsilon > 0$ s.t. $A_p \exp_q|_{B_{\varepsilon}(0)}$. $v \in K$.

Look at $c$ at time $p - \varepsilon/2$.

Thus contracts def of $p$, so have const. a geod of length $p + \varepsilon/2$.
Lecture 17: Last topic: short geodesics minimizing length

Today: \[ \text{Length } (S_r(p)) = 2\pi r \left( 1 - \frac{K(p)}{6} r^2 + O(r^3) \right) \]

**Geodesic polar coord:** Fix orthonormal basis \( e_1, e_2 \) for \( T_pS \).

\[ f : (0, R_0) \times (0, 2\pi) \to \mathbb{R}^2 \]

\[ \exp_p (r \cos \theta e_1 + r \sin \theta e_2) \]

\[ g_{11} = 1, \quad g_{21} = g_{22} = 0, \quad g_{22} = g(r, \theta). \]

**[constant]**

\[ \text{length } (S_r(p)) = \int_0^{2\pi} \frac{1}{\sqrt{g(r, \theta)}} \, d\theta \]

**Calculating \( K(r, \theta) \)**

\[ \{ f_r, f_\theta, n \} \]

- \( f_{rr} = \frac{\partial g}{\partial r} - L_{11} n \)
- \( f_{\theta r} = \frac{1}{2} \frac{\partial g}{\partial r} f_\theta - L_{21} n \)

Look at \( f_{rr\theta} = f_{\theta rr} \)

\[-L_{11} L_{22} = \frac{1}{2} (\frac{\partial g}{\partial r})^2 r + \frac{1}{4} (\frac{\partial g}{\partial r})^2 - L_{21} L_{12} \]

\[ \Rightarrow -K = -\det (L_{ij}) = \frac{(\sqrt{6^2}) r r}{\sqrt{6}} \]

\[ \text{should be } L_{12} \]
Fix $\Theta_0$, consider $\alpha_{\Theta_0}(r): (-R_0, R_0) \rightarrow \mathbb{R}$
given by $\alpha_{\Theta_0}(r) = \sqrt{6(r, \Theta_0)}$ for positive $r$

$V = \cos \Theta_0 e_1 + \sin \Theta_0 e_2$
$W = -\sin \Theta_0 e_1 + \cos \Theta_0 e_2$

$$\alpha_{\Theta_0}(r) = \left| D_{rv} \exp_p (rw) \right| = r \left| D_{rv} \exp_p (w) \right|$$

Note: These two agree,

$G(r, \Theta_0) = D_{(r, \Theta_0)} f(0, 1)$

and $f = \exp \circ \rho$

$\rho \circ \Theta_0 = D_{(r, \Theta_0)} f(0, 1) = rw$

However, $\alpha_{\Theta_0}(r)$ is smooth as $= r \left| D_{rv} \exp_p (w) \right|$ involves $\sqrt{\cdot}$ but only near $1$.

$$\alpha_{\Theta_0}(0) = 0$$

$$\alpha_{\Theta_0}'(0) = \left( \left| D_{v} \exp_p (w) \right| + r \left| D_{v} V \right| \right) \big|_{r=0} = 1$$

$$\alpha_{\Theta_0}''(0) = \lim_{r \searrow 0} \left( \alpha_{\Theta_0}''(r) = \sqrt{6(r, \Theta_0)} r = -K(r, \Theta_0) \alpha(r) \right) = 0$$

$$\alpha_{\Theta_0}'''(0) = \lim_{r \searrow 0} \left( \alpha_{\Theta_0}'''(r) = -K(r, \Theta_0)' \alpha(r) = K(r, \Theta_0) \alpha'(r) \right) = -K(\rho)$$
So: \( \alpha_\theta (r) = r - \frac{K(p)}{6} r^3 + O(r^4) \) \\
Recall \( f(r) \) in \( O(r^4) \) \\
if \( \exists C \) s.t. \( |f(r)| \leq C r^4 \)

Length \( (S_r(p)) \) = \( \int_0^{2\pi} \alpha_\theta (r) \, d\theta = 2\pi r \left(1 - \frac{K(p)}{6} r^2 + O(r^3)\right) \)

[Query: how did he cheat?] \( \alpha_\theta (r) = r - \frac{K(p)}{6} r^3 + E_\theta (r) \)

\( \text{Aside: } f(x,y) = \begin{cases} \frac{xy^2}{x^2 + y^4} & x \neq 0 \\ 0 & x = 0 \end{cases} \)

Along any ray \( f(r, \theta_0) = O(r) \)

\( f(y^2, y) = \frac{1}{2} \) finally

\( E_\theta (r) = \frac{1}{4!} \alpha_\theta^{(4)} (r_0) \)

\( r_0 \in (0, r) \)

\( \alpha_\theta^{(4)} (r_0) = -2 K(r, \theta) r \)

Think of \( K(\exp_p) \) as a fun of a pt in \( T_p S \)

On a cusp set, any directional derivative in a unit direction is bounded

\( K(r, \theta)_r = a K(r, \theta) e_1 + b K(r, \theta) e_2 \quad v = a e_1 + b e_2 \)

This error term is uniformly under control

so formula for len. holds. So Gauss curvature is intrinsic. \( \Box \)
Today: Gauss-Bonnet: [local and global.]

Local version: geodesic triangles.

$K > 0$

$\Sigma \theta_i = \frac{3}{2} \pi$ for unit $\Delta,$

$\Sigma \theta_i > \pi$

$K = 0$

$\Sigma \theta_i = \pi$

$K < 0$

$\Sigma \theta_i < \pi$

**Thm:** $S \subseteq \mathbb{R}^3$ smooth surface. $T$ in $S$ a geod. $\Delta.$

with interior angles $\theta_1, \theta_2, \theta_3.$ Then

$$\Sigma \theta_i - \pi = \int_T K \, dA$$

[Point out reasonable w.r.t. scaling.]

Where $\int$ means: Suppose $f: U \to S$ w/ $f(U) = T$

$$\text{Area}(T) = \iint_{f^{-1}(T)} \sqrt{g_{ij}} \, dx \, dy \quad [g_{ij} \text{ metric coeffs}]$$

Suppose $\rho: S \to \mathbb{R}$ some cont fn

$$\int_T \rho \, dA = \iint_{f^{-1}(T)} \rho \circ f \sqrt{g_{ij}} \, dx \, dy$$
For regions not contained in a slant, break into pieces integrate over each piece, add the result. [as w/ area, doesn't matter how we subdivide.]

**Gauss-Bonnet:** Suppose $S \subseteq \mathbb{R}^3$ is a compact smooth surface. Then $\int_S K \; dA = 2\pi \chi(S)$

[Discuss why this is surprising.]

**Cor.** Suppose $S$ is a ept surface in $\mathbb{R}^3$ w/ $K > 0$ everywhere. Then $S \cong S^2$

**Pf:** Gauss-Bonnet + P doesn't embedd in $\mathbb{R}^3$.

---

**Pf that local $\Rightarrow$ global.**

A triangulation of $S$ is geodesic if every edge is a geodesic segment. \(\text{Ex.}\)

**Thm 8.4.1:** Any ept surface $S$ in $\mathbb{R}^3$ has a geodesic triangulation.

[constructed locally at small scales]
Let $T_1, \ldots, T_n$ be the triangles of a good tri of $S$

$$\int_S K \, dA = \sum_{i=1}^{n} \int_{T_i} K \, dA = \sum_{i=1}^{n} \left( \Theta_{i,1} + \Theta_{i,2} + \Theta_{i,3} - \pi \right)$$

$$= \sum \text{ (interior angles)} - \pi n$$

$$= 2\pi \text{ (# of verts)} - \pi \text{ (# triangles)}$$

$$= 2\pi \left( V - \frac{1}{2} f \right) = 2\pi \chi(S)$$

[Query] Each triangle contributes 3 edges, double counting, so $e = \frac{3}{2} f$

$$V - e + f = V - \frac{1}{2} f$$

Proof of 8.4.1: Key idea: $B = \exp_p (B_\varepsilon(0))$ is geodesically convex for small $\varepsilon$, i.e. any two pt in $B$ are joined by a unique minimal geod arc which lies in $B$. Any two such geod intersect in at most one pt. Then cover $S$ w/ geod polygons. Then triangulate the comp regions, as in Euclidean space.
Lecture 19: Last time: Gauss-Bonnet: $\int_S K dA = 2\pi\chi(S)$

Follows from: Then: $T \subseteq S$ a good triangle. Thus, $\int_T K dA = \sum \Theta_i - \pi$

Today: Proof of Thm.

Assume that $T$ is small enough to be contained in a good, polar coor patch. [Otherwise, stop up.]

Let $f: (0, R_0) \times (-\delta, 2\pi - \delta) \rightarrow S$

$f(r, \theta) = \exp_p(r(\cos \theta e_1 + \sin \theta e_2))$

$\tilde{u} = f^{-1}(u), \tilde{v} = f^{-1}(v)$

Claim 1: $\tilde{c}$ intersects each line $(x, \Theta)$ in at most one pt.

Suppose not. By IVT, $f$ a point where $\tilde{c}$ is horizontal. [Query] contradicts uniqueness of geodesics

Thus, $\tilde{c}$ can be param by $(h(\theta), \theta)$ for $\theta \in (0, \Theta_0)$.

$\int_T K dA = \int_0^\Theta \int_0^{h(\theta)} K(f(r, \theta)) \sqrt{\det(g_{ij})} \, dr \, d\theta$

$g_{11} = 1$
$g_{12} = g_{21} = 0$
$g_{22} = G$

$-K = \frac{\sqrt{G}}{\sqrt{G}}$
\[
\begin{align*}
\int_0^\Theta \int_0^{h(\theta)} -\sqrt{G} \, rr \, dr \, d\theta &= \int_0^\Theta \sqrt{G} \, r \, d\theta \\
&= \int_0^{\Theta_1} 1 - (\sqrt{G})_r (h(\theta), \theta) \, d\theta = \Theta_1 + \int_0^{\Theta_2} - (\sqrt{G})_r (h(\theta), \theta) \, d\theta \\
&\quad \text{(at least, this looks good!)}
\end{align*}
\]

Define \( \phi(\theta) \) to be \( \phi(\theta) \) in S.

Claim: \( \phi'(\theta) = - (\sqrt{G})_r (h(\theta), \theta) \)

Assuming this get

\[
\int_T K \, dA = \Theta_1 + \phi \bigg|_0^{\Theta_2} = \Theta_1 + \phi(\Theta_1) - \phi(0) = \Sigma \Theta_i - \pi
\]

Choose \( \alpha : [0, \varepsilon] \rightarrow [0, \Theta_0] \) so that

\[
\begin{align*}
\varepsilon(\theta) &= (h(\theta), \theta) \\
c &= f \circ \varepsilon
\end{align*}
\]

\( \varepsilon \circ \alpha(s) = (h \circ \alpha(s), \alpha(s)) \)

Geodesic equation (using \( \Gamma_{ij}^k \) from HW) for 1st con give:

\[
(h \circ \alpha)'' = \frac{1}{2} G_{ij}(\varepsilon \circ \alpha(s))(\alpha'(s))^2
\]
Let's compute $\phi$:

\[
\cos (\phi \alpha (s)) = f_r \cdot (\cos)^' = (h \cos') (s)
\]

\[
\sin (\phi \alpha (s)) = \cos (\pi/2 - \phi \alpha (s)) = \frac{f_0}{\sqrt{6}} \cdot (\cos)^' = \sqrt{6} \alpha' (s)
\]

Thus

\[
\frac{1}{2} G_r(\cos (s)) (\alpha' (s))^2 = (h \cos'' (s)) (s)
\]

\[
= (\cos (\phi \alpha (s))^' (s) = -\sin (\phi \alpha (s)) \phi' (\alpha (s)) \alpha'' (s)
\]

\[
= -\sqrt{6} (\cos (s)) \phi' (\alpha (s)) (\alpha'' (s))^2
\]

\[
= -\frac{1}{2} \frac{G_r (h \alpha (s), \alpha (s))}{\sqrt{G(h \alpha (s), \alpha (s))}} = -\sqrt{6} r (h \alpha (s), \alpha (s))
\]

as desired.

Q.E.D.
Lecture 20: Geometry of abstract surfaces.

[Problematic things:] Nice: $\int_{S^2} K \, dA = 4\pi$ also $\exists$ a sphere w/ $K = 1$ everywhere

Bad: $\int_{T^2} K \, dA = 0$ but D.N.E. a $T^2$ in $\mathbb{R}^3$ w/ $K = 0$ everywhere.

Also $K = 0$

but us closed surface of curv. $K = -1$ [Holed]

Def. Let $S \subseteq \mathbb{R}^n$ be a smooth surface. A Riemannian metric $I$ on $S$ is a family of pos def sym bilinear forms

$I_p : T_p S \times T_p S \to \mathbb{R}$ which is smooth.

$s$-smooth means: A coor chart $f : U \to S$ but

$g_{ij}(p) = I_p(f_i(p), f_j(p))$. Then $g_{ij}$ is a smooth fn.

Ex: Hypaloid model for the hyperbolic plane.

$x, y \in \mathbb{R}^3$, $\langle x, y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3$

$H^2 = \{x \in \mathbb{R}^3 | \langle x, x \rangle = -1, x_3 > 0\}$

$I_{x_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$\langle x, x \rangle = 0$ $\quad x_1^2 = x_3^2 \Rightarrow x_1 = \pm x_3$

$\langle x, x \rangle = -1$ $\quad x_1^2 - x_3^2 = -1$
Topologically this is just $\mathbb{R}^2$.

For $p \in \mathbb{H}^2$, define $I_p : T_p \mathbb{H}^2 \to \mathbb{R}$ by $I_p(v, w) = \langle v, w \rangle$ - Lorentzian.

[Query: What do we need to check to see this is Riemannian?]

Consider $p_0 = (1, 0, 0)$. Then $I_{p_0}$ is usual Euclidean inner prod on $\mathbb{R}^2$ do we ok here.

Set $O(2, 1) = \{ A \in GL_3(\mathbb{R}) \mid \langle Ax, Ay \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathbb{R}^3 \}$

["isometries of $\langle , \rangle" ]

$O_0(2, 1)$ those which preserve $\mathbb{H}^2$

$SO_0(2, 1)$ those w/ det 1.

Claim: $SO_0(2, 1)$ acts transitively on $\mathbb{H}^2$, i.e. given $x, y \in \mathbb{H}^2$ $\exists A \in SO_0(2, 1)$ such that $Ax = y$.

Cor: $I_p$ is always pos def $\Rightarrow (\mathbb{H}^2, I_p)$ is a Riemannian surface.

Pf: Any $M \in O(2)$ gives an element via $\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & X_3 & X_2 \\ 0 & X_2 & X_3 \end{pmatrix}$ takes $(0, 0, 1)$ to $(0, X_2, X_3)$

\[ X_3X_2 - X_2X_3 = 0 \]
\[ 1, X_3^2 - X_2^2 = 1 \]
Note: $A \in O_0(2,1)$ gives an isometry of $(\mathbb{H}^2, \mathbb{I}_p)$.

Def: Any intrinsic notion encountered earlier is defined for a Riemannian surface in the same way (usually via local coordinates).

Geodesics in $\mathbb{H}^2$: $\gamma = (\text{Plane through } 0) \cap \mathbb{H}^2$

Pf: Since $e^t A \in O_0(2,1)$ take planes (through $0$) to planes (through $0$) and it acts transitively; it is enough to check this for planes through $p_0$.

But this is clear using the reflection symmetry.

Exercise: We can move any pt to any other via an isometry, $K =$ constant

\[ K > 0 \text{ implausible; } K = 0 \text{ should imply that } \mathbb{H}^2 \text{ is isometric to } \mathbb{R}^2, \text{ but it violates the parallel postulate.} \]

HW: cite fact $K = -1$. 
Another point of view: Poincaré Disc Model

\[ D = \{ z \in \mathbb{C} \mid |z| < 1 \} \]

\[ \overline{P} (v, w) = \frac{4}{(1 - r^2)^2} \]

Angles same but dist. distorted

Geodesics:

Note: Boundary is infinitely far away

Note: Is isometric to earlier example.
Lecture 21: Hyperbolic plane

Poincaré Model: \( D = \{ |z| < 1 \mid |\bar{z}| < 1 \} \)  

Hyperbolic model (last time) 

Poincaré disc model (today) 

Usual Euclidean dot product:

\[ I_{p}(v, w) = \frac{4}{(1-r^2)^2} \sqrt{v \cdot w} \]

\[ r = \sqrt{x^2 + y^2} \]

Distances are distorted but not angles. Boundary is infinitely far away.

Geodesics: circles meeting boundary in right angles (w/ straight lines through \( 0 \) as a special case.)

Can prove these are geodesics through symmetry, but understanding the others will take some work.

Inversions:

\( i_{C} : \hat{C} \rightarrow \hat{C} \) 
\( \hat{C} = C \cup \{ \infty \} = CP' = S^2 \)

\( i_{C}(p) \)

Also (center \( \leftarrow \rightarrow \infty \))

Prop: \( i_{C} \circ i_{C} = id \)
Lemma: Under $i_c$: *circles not containing the center of $C$ go to circles not containing the center*.

- circles through the center go to lines
- lines go to circles or circles.

Proof: Check alg, starting w/unit circle about 0, noting

\[ i_c = (z \mapsto 1/z) \] and using complex notation, e.g.

\[ C = \{ z \mid |z - c_o|^2 = r_o^2 \}. \]

Lemma: $i_c$ preserves angles (is conformal)

Proof:

\[ \theta \]

Back to Poincaré model:

If $C \perp \partial D$, then $i_c$ preserves $D$.

Claim: $i_c$ is an isomor of $(D, \Gamma_p)$

(\[
\rightarrow \text{geodesics are as claimed}\]

Good sign: $i_c$ preserves angles.
Pf. of Claim: Calculation

Plausibility Argument:

\[ \sqrt{I_p(\text{Euc. unit})} = \left( \frac{dL}{d\alpha} \right)^{-1} = \frac{2}{1-r^2} \]

\[ \text{Probally skip} \]

Facts:

\( \text{Isom}_0^1(D; I_p) \) is generated by inversions.

\( \text{Isom}_0^1(D; I_p) = \) biholomorphic maps from \( D \) to itself

\( \text{orientation preserving} = \left\{ z \mapsto e^{i\theta \left( \frac{z - \alpha}{-\alpha z + 1} \right)} \mid \theta \in \mathbb{R}, \alpha \in D \right\} \)

This is really the same as what we looked at last time — \( \text{Isom}_0^1(D; I_p) \) is transitive and it can't be the Euclidean plane because it violates the parallel postulate:

Explicitly: Construct a map using geodesic polar coos.
Triangles: $\int \kappa dA = \Theta_1 + \Theta_2 + \Theta_3 - \pi$

$\Rightarrow \text{Area} = \pi - \Theta_1 - \Theta_2 - \Theta_3 \Rightarrow \text{Every triangle has area less than } \pi!$

Ideal triangles: all "vertices" at $\infty$

These are all isometric for the following reason.

Take one edge to

invert in some circle like this

These all have area $\pi$ (HW).
Lecture 22: Last time: Poincaré model.

Today: Tilings of hyperbolic plane, hyperbolic metric on elliptic surfaces.

Ideal triangles:

\[ I_p(v,w) = \frac{4}{(1-r^2)^2} \cdot v \cdot w \]

Can do this symmetrically, i.e.

Given two triangles \( T_1 \) and \( T_2 \), there exists \( g \in \text{Isom}(H^2) \) taking \( T_1 \) to \( T_2 \), preserving the whole tiling.

The one I drew is not symmetric, hand out one that is.

Query to nature of the difference.

Issue: isometric to \( \mathbb{R} \)

Upper Halfspace Model. \( H = \{ z \in \mathbb{C} \mid \text{Re} \, z > 0 \} \)

An isometry:

\[ p = x + iy \]

\[ \begin{align*}
D & \rightarrow H \\
z & \rightarrow \frac{z + i}{iz + 1}
\end{align*} \]

Geodesics are still circles, meeting \( \partial \) in \( \mathbb{R} \).
\[ \text{Isom}^+(H) = \left\{ z \mapsto \frac{az + b}{cz + d} \mid a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\} \]

can rescale all simult. without changing the result of Möbius transformation.

\[ \text{PSL}_2 \mathbb{R} = \frac{\text{SL}_2 \mathbb{R}}{\{ \pm I \}} \cong \text{Isom}^+(H) \]

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left( z \mapsto \frac{az + b}{cz + d} \right)
\]

What does this have to do with tilings?

\[ \Gamma = \text{PSL}_2 \mathbb{Z} \leq \text{PSL}_2 \mathbb{R} \]

Preserves this tiling:

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2 \mathbb{Z} \]

Apply to \( \frac{p}{q} \in \mathbb{Q} \)

\[
\frac{p}{q} \mapsto \frac{ap + bq}{cp + dq} = \frac{r}{s} \quad \text{where} \quad (r) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(p)
\]

\[ \frac{p}{q} \longleftrightarrow (p, q) \]
Now draw a good from $P_1q_1$ to $P_2q_2$ if $|P_1q_2 - P_2q_1| = 1$.

This is preserved by $A$ as

\[
\text{det} \begin{pmatrix} P_1 & P_2 \\ q_1 & q_2 \end{pmatrix} \quad \text{and}
\]

taking \( \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = A \begin{pmatrix} p_1 \\ q_2 \end{pmatrix} \) gives the desired result.

This is the triangulation shown above. (Handout has the same in the disc model.)

Mention connection to modular forms, # theory etc.

What about bounded tiles?

**Lemma:** There is a right angle pentagon in $H^2$.

**Pf:** Use disc model.

Consider the pentagon $P(r)$ shown, and consider $\Theta(r)$

For small $r$, $P(r)$ is nearly Euclidean, hence $\Theta(r) = 2\pi/5$

Also $\Theta(1) = 0$. By continuity, there $\exists r$ with $\Theta(r) = \pi/2$
Can tile $\mathbb{H}^2$ with such pentagons, in a nested, symmetric fashion. See Handout. To prove:

1) Write down group explicitly.

2) Use:

**Theorem:** Let $I_p$ be a Riemannian metric on $\mathbb{R}^2$ such that $K = -1$ everywhere. If $(\mathbb{R}^2, I_p)$ is complete in its intrinsic metric, then $(\mathbb{R}^2, I_p)$ is isometric to $(\mathbb{H}^2, I_p)$. and assemble pieces locally into a plane which has such a Riemannian metric.

What about compact surfaces?

Use regular hyp. octagon w/ vertex angles $\pi/4$. Gives nice metric on $S$ w/ $K = -1$ everywhere. Then $\tilde{S}$ univ. cover has a Riemann metric making it into $\mathbb{H}^2$! In particular, $\tilde{S} \cong \mathbb{R}^2$.

See Handout for induced tiling.
Lecture 23: Last time: Tiling of $H^2$

Today: Hyperbolic metrics on closed surfaces

Back to topology: Homology.

What about compact surfaces? \[ \int_S K \, d\alpha = 2\pi \chi(S) \]

$\chi < 0$, suggests a metric of constant curve $< 0$.

glue sides by isometries (ob as have same length)

metric makes sense around vertex as angles add to $2\pi$.

$\pi_1 \cong \mathbb{Z}^2$ of translations

Euclidean case:

\[ \mathbb{R}^2 / \mathbb{Z}^2 \]

local isometry

Hyperbolic case:

\[ S = \]

Use regular octagon in $H^2$ w/ vertex angles $\frac{\pi}{4}$.

Gives a metric on $S$ w/ $K = -1$ everywhere.

Then $\tilde{S}$ has a Riemann metric making it into $H^2$.

In particular, $\tilde{S} \cong \mathbb{R}^2$!

see handout for picture.
Works same for any surface with $X(S) < 0$.

In general, can change any Riemannian metric into a constant curvature one:

**Uniformization Theorem:** Let $S$ be a closed surface with $\chi$-metric $I_p$.

Then $\exists$ a smooth function $\phi: S \to \mathbb{R}_{>0}$ such that $\phi(p)I_p$ is a Riemannian metric of constant curvature $+1, 0,$ or $-1$.

Study of constant curvature metrics $\leftrightarrow$ complex structures.

Many on $T^2$: 

- Modular space of curves

$M(S) = \text{constant curvature metrics on } S$ up to isom

$J(S) = \text{constant curvature metrics on } S$ remembering how the surface "wears" them.

$S = \hat{\omega}$

$H^2 / \text{PSL}_2 \mathbb{Z} = \sqrt{H^2}$

$S = \hat{T} \# \cdots \# \hat{T}$ complicated

$g \geq 2$

Mapping class group.
And now for something completely different...

[easy to compute, but hard to tell]

Fundamental group - [answers apart]

Measures only "1-dimensional" part of X. Can't tell $S^3$ from $S^4$.

[Need invariants that measure homology in higher dimensions.]

$\pi_1 X$ is about maps $S^1 \to X$ [Query] $S^n \to X \pi_n X$

"Higher homology groups."

All $\pi_n S^2$ are not known.

Homology:

$H_n(X) = n$ dim'l things w/o boundary

$\begin{align*}
n = 0 & \quad n = 1 & \quad n = 2 \\
\text{w/o} & \quad \emptyset & \quad \circ
\end{align*}$

Boundaries of $n+1$ dim'l things.

$\begin{align*}
n/a & \\
\text{boundaries} & \quad n/a
\end{align*}$

$\begin{align*}
\text{same in } H_0
\end{align*}$

[Eventually, will define $H_n(X)$ for any topological space.]

$K$ a simplicial complex [Query: finite # of simplices]

in $\mathbb{R}^n$
$C_0(X) = \mathbb{Z} \oplus \mathbb{Z} = \{a_0 v_0 + a_1 v_1\}$

$C_1(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = \{c_1 e_1 + c_2 e_2 + c_3 e_3\}$

$C_n(X) = 0$ for $n > 1$.

**Boundary map**: $\partial_n : C_n(X) \to C_{n-1}(X)$ a homomorphism

$\partial_1 : C_1(X) \to C_0(X)$

$\partial_1(e_1) = v_1 - v_0$

$\partial_1(e_2) = v_1 - v_0$

**Cycle**: $c \in C_n(X)$ s.t. $\partial_n c = 0$, $\ker \partial_n$

0-cycles: $\ker \partial_0 = C_0(X)$

[ $n$-dim'l things w/o boundary,]

1-cycles: $\partial_1(c_1 e_1 + c_2 e_2 + c_3 e_3) = (c_1 + c_2 + c_3)(v_1 - v_0)$

$\ker \partial_1 = \text{those } \omega / c_1 + c_2 + c_3 = 0$

**Def**: $H_n(X) = \ker \partial_n / \text{im} \partial_{n+1}$

$H_0(X) = C_0(X) / \text{im} \partial_1 = C_0(X) / \langle (v_1 - v_0) \rangle$

$= \mathbb{Z} / \langle (1, -1) \rangle = \mathbb{Z}$

$H_1(X) = \ker \partial_1 / \text{im} \partial_2 = \ker \partial_1 = \mathbb{Z}^2 \omega / \text{basis } c_1 - c_2 c_2 - c_3$

3 cycles: not a cycle.

3 cycles: change of basis.

3 cycles: not a cycle.
**Def:** An oriented simplex is one with an ordering of the vertices \([V_0, ..., V_n]\).

Fix an orientation on all simplices by ordering all the vertices of \(K\).

\(n\)-chains: \(\mathcal{C}_n(K) = \bigoplus_{ \text{all} \text{ free abelian gp w/ basis the n-simp of } K} \mathbb{Z} \)

**clm ex:**
\(\mathcal{C}_0(K) = \mathbb{Z}^4 = \{a_0V_0 + a_1V_1 + a_2V_2 + a_3V_3\}\)

\(\mathcal{C}_1(K) = \mathbb{Z}^5\) basis: \(e_1 = [V_0, V_1]\) etc.
\(\mathcal{C}_2(K) = \mathbb{Z}\) basis: \(e_1 = [V_0, V_1, V_2]\)

**Boundary maps:** \(\partial_n: \mathcal{C}_n(K) \rightarrow \mathcal{C}_{n-1}(K)\) a homomorphism.

\(n=0, \quad \partial_0 = 0\)

\(n=1, \quad \partial [w_0, w_1] = w_1 - w_0, \quad \text{e.g. } \partial e_1 = V_1 - V_0\)

\(n=2, \quad \partial \sigma = [V_1,V_2] - [V_0,V_2] + [V_0,V_3] = V_0 e_3 - e_2 + e_1\)
\[ \partial_n ([w_0, \ldots, w_n]) = \sum_{i=1}^{n} (-1)^i [w_0, \ldots, \hat{w_i}, \ldots, w_n] \parallel [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n] \]

**n-cycles:** \( \text{ker} \partial_n \) "Things w/o boundary."

**Ex:** \( n=0 \) just \( C_0(K) \)

\( n=1 \)

\[ \partial (b_1 e_1 + \ldots + b_5 e_5) = (-b_1 - b_2) v_0 + (b_1 - b_3 - b_5) v_1 + \ldots \]

If \( 0 \) have each con \( 0 \), i.e. \( b_1 = b_3 + b_5 \)

**Basis for \( \text{ker} \partial_n \):**

\( \mathbb{Z}_1 \quad \mathbb{Z}_2 \)

**n-boundaries:** \( \text{im} \partial_{n+1} \) "boundaries of \( n \) dim'l things"

\( n=0 \) \( \text{im} \partial_1 = \{ \sum a_i v_i \mid \sum a_i = 0 \} \)

\( n=1 \) \( \text{im} \partial_2 = \mathbb{Z}_1 \)

\( n \geq 2 \) \( 0 \)
Lemma: \( \partial_n \circ \partial_{n+1} = 0 \) \( \Rightarrow \) \( \ker \partial_n \supset \text{im} \partial_{n+1} \). 

Thus: \( \text{Set } H_n(K) = \frac{\ker \partial_n}{\text{im} \partial_{n+1}} \).

In our example:

\[ H_0(K) = \mathbb{Z} \quad \text{in general } \cong \mathbb{Z}^{\# \text{ of conn. comp}} \]
\[ H_1(K) = \mathbb{Z} \quad \text{in general } \cong \prod_{i=0}^{\text{dim}(\partial)} (|K|) \]
\[ H_2(K) = 0 \]

Proof of lemma: Suffices to check it on basis elements

\[ \partial_n (\partial_{n+1} ([w_0, \ldots, w_{n+1}])) = \partial_n \left( \sum_{i=0}^{n+1} (-1)^i [w_0, \ldots, \hat{w}_i, \ldots, w_{n+1}] \right) \]

\[ = \sum_{i=0}^{n+2} (-1)^i \left( \sum_{j=0}^{i-1} (-1)^j [w_0, \ldots, \hat{w}_j, \ldots, w_{n+1}] + \sum_{j=i+1}^{n+2} (-1)^{i-1} [w_0, \ldots, \hat{w}_i, \ldots, \hat{w}_j, \ldots, w_{n+1}] \right) \]

\[ = 0 \quad \text{as each term appears twice with opposite signs}. \]
Example:

\[ H_0 = \mathbb{Z} \]
\[ H_1 = 0 \]
\[ H_2 = \mathbb{Z} \]

Fact: \( H_n \) only depends on \( |K| \)

\[ H_n S^n = \mathbb{Z} \]
Lecture 25: Last time: $H_n(K)$ for a simplicial complex.

Today: Maps of spaces induce maps on $H_n$.

- Why $H_n$ only depends on $|K|$.

**Did Not actually use**

---

$f: K_1 \rightarrow K_2$ a simplicial map

\[ \text{or} \quad \circlearrowleft \]

Each simplex is mapped onto a simplex linearly.

\[ f^*_n: C_n(K_1) \rightarrow C_n(K_2) \]

\[ \sigma \rightarrow \begin{cases} \pm f(\sigma) & \text{if } f(\sigma) \text{ is an } n \text{-simplex} \\ 0 & \text{otherwise} \end{cases} \]

Where the sign is the sign of the permutation:

\[ \sigma = [v_0, \ldots, v_n] \quad \quad [w_0, \ldots, w_n] \rightarrow [f(v_0), \ldots, f(v_n)] \]

\[ f(\sigma) = [w_0, \ldots, w_n] \]

**Key:** $f_\#$ is a chain map: $f_\# \circ \partial_n = \partial_n \circ f_\#$

Chain complex:

\[
\begin{align*}
C_{n+1}(K_1) &\rightarrow C_n(K_1) \xrightarrow{\partial_n} C_{n-1}(K_1) \\
C_{n+1}(K_2) &\rightarrow C_n(K_2) \xrightarrow{\partial_n} C_{n-1}(K_2)
\end{align*}
\]
Consequences: \[ f_\#(\ker \partial_n^{K_1}) \leq \ker \partial_n^{K_2} \]
\[ f_\#(\text{im } \partial_{n+1}^{K_1}) \leq \text{im } \partial_{n+1}^{K_2} \]

\[ \Rightarrow H_n(K_1) = \frac{\ker \partial_n^{K_1}}{\text{im } \partial_{n+1}^{K_1}} \xrightarrow{f_*} H_n(K_2) = \frac{\ker \partial_n^{K_2}}{\text{im } \partial_{n+1}^{K_2}} \]

Reason for key:
\[ \sigma \xrightarrow{f} \sigma \]

\[ f_\#(\sigma) \]

\[ \sigma f(\sigma) \]

\[ \sigma f(\sigma) \]

\[ \sigma f_\#(\sigma) \]

\[ f_\#(\sigma) = -f(\sigma) \]

General case is the same, breaking permutation into a product of transpositions. (K_{k+1}).

[see Armstrong for details.]

**Thm:** Suppose \( f, g : K_1 \rightarrow K_2 \) are homotopy maps which are simplicial. Then \( f_* = g_* : H_n(K_1) \rightarrow H_n(K_2) \)

**Thm:** Suppose \( f : K_1 \rightarrow K_2 \) is any map, then it is homotopic to a simplicial one on some subdivision \( K'_1 \) of \( K_1 \).
Lecture 25:

Introduced singular homology.

gave map $H_n^\Delta (K) \rightarrow H_n^{\text{Singular}} (|K|)$

including the def of chain map.

Followed pages 22/23 of 2004 151a notes.
Lecture 26: Last time: Singular Homology

Today:

Let \( X \) be a topological space. Then \( C_n(X) \) is a free abelian group with basis all \( \sigma: \Delta^n \to X \).

\[
H_n(X) = \ker \partial_n / \text{im} \partial_{n+1}
\]

\[
\partial_n \sigma = \sum (-1)^i \sigma |_{[e_0, \ldots, \hat{e_i}, \ldots, e_n]}
\]

If \( f: X \to Y \) have \( \pi_1 X \to \pi_1 Y \); similarly \( H_n(X) \to H_n(Y) \)

given by:

\[
f_*: C_n(X) \to C_n(Y)
\]

\[
(\sigma: \Delta^n \to X) \to f_*(\sigma)
\]

This is a chain map, i.e.

\[
f_* \circ \partial_n = \partial_{n+1} \circ f_*
\]

So get \( H_n(X) \to H_n(Y) \) [Also have in simplicial homology]

Note: If \( X \to Y \to \mathbb{Z} \) then \( (g \circ f)_* = g_* \circ f_* \)

as \( (g \circ f)_*(\sigma) = g \circ f \circ \sigma = g \circ (f \circ \sigma) = f_* (g_*(\sigma)) = f_*(g_*(f_*(\sigma))) \).

Lemma: If \( f, g: X \to Y \) are homotopic maps, then

\[
f_* = g_*: H_n(X) \to H_n(Y).
\]

Pf: See Hatcher.
Of idea:

\[ Z = s_1 + s_2 + s_3 \]

see that \( \exists C \in s.t. \quad \partial_2 C = f_# z - g_# z \).

More schematic:

\[
\begin{align*}
X & \xrightarrow{f_#} Y \\
\Rightarrow & \xrightarrow{g_#}
\end{align*}
\]

Thm: Suppose \( X \xleftarrow{g} Y \) are inverse homotopy equiv. \( (f \circ g \approx \text{id}_Y) \) \( (g \circ f \approx \text{id}_X) \).

Then \( H_n(X) \xrightarrow{f_*} H_n(Y) \) is an isomorphism for all \( n \).

Pf: \( H_n(X) \xleftarrow{f_*} H_n(Y) \)

by Lemma.

\( g_# f_* = (g \circ f)_* = (\text{id}_X)_* = \text{id}_{H_n(X)} \).

Same for \( f_* \circ g_* \Rightarrow f_* \) and \( g_* \) are inverse isom.
Case: If $X$ is contractible, e.g. $\mathbb{R}^k$, then $H_n(X) = H_n(pt) = \begin{cases} \mathbb{Z} & n=0, \alpha_k \\ 0 & n>0 \end{cases}$

Lemma: $H^*(S^k = \{ x \in \mathbb{R}^{k+1} \mid |x| = 1 \}) = \begin{cases} \mathbb{Z} & n = 0 \vee k \\ 0 & \text{otherwise} \end{cases}$

[Draw for $S^2$, $S^3$ on HW in terms of $H_n$]

Key: $A \subset X$, $A$, $X$ path connected, $[A \ ""\text{retractable}""]$

Can relate $H_n(A)$, $H_n(X)$, $H_n(X/A)$ via $A \rightarrow X \rightarrow X/A$

$\rightarrow H_{n}(X/A) \xrightarrow{id} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X/A) \xrightarrow{\partial} H_{n-1}(A)$

is exact, i.e. at any term $A \rightarrow T \rightarrow V$

we have $\text{im}(a) = \text{ker}(b)$. [Know two can usually deduce the 3rd]

$B^n = n\cdot \text{dim' ball}$

$S^{n-1} \rightarrow B^n \rightarrow B^n/S^{n-1} = S^n$
**Cor:** $\mathbb{R}^n \neq \mathbb{R}^m$ if $n \neq m$

**Pr:** $\mathbb{R}^n \setminus \{pt\} \subset S^{n-1}$.

*Thm:* Any map $f: B^n \to B^n$ has a fixed pt, i.e. an $x_0$ s.t. $f(x_0) = x_0$.

**Pr:** Suppose $f$ lacks such a fixed pt.

\[ r: B^n \to S^{n-1} \text{ continuous} \]

\[ r|_{S^{n-1}} = id \quad \text{retract of} \]

\[ H_n(S^{n-1}) \xrightarrow{i_*} H_n(B^n) \xrightarrow{r_*} H_n(S^{n-1}) \]

\[ id = (r \circ i)_* = r_* \circ i_* = 0 \]
Lecture 27: The final installment.

The Euler Characteristic of a simplicial complex

\[ K \text{ is } X(K) = \sum_{n=0}^{\infty} (-1)^n (\# \text{ of } n\text{-simplices}) \]

[Also makes sense for \( \Delta \)-complexes w/ finitely many cells]

[agrees with notion for surfaces]

Thm. Let \( K_1, K_2 \) simplicial complexes. If \( |K_1| \cong |K_2| \), then \( X(K_1) = X(K_2) \).

Thus provided \( X \) has some triangulation, then it makes sense to talk about \( X(X) \).

When \( X \) is a surface already have 1.75 proofs:

1) On 1st HW, via classification

[Query:] 2) Gauss-Bonnet: \( \oint_S K \cdot dA = 2\pi X(\text{a good triangulation}) \)

View as fixed

\[ X(K) = \sum (-1)^n \text{ (rank of free part of } H_n(|K|)) \]

\[ \mathbb{Z} \oplus \text{ finite} \]

\[ \text{singular homology} \]

\[ \text{ torsion is annoying, would be nice if homology gaps are vector spaces} \]
Homology w/ coefficients: \( F \) a field \([\mathbb{Z}/2 \text{ or } \mathbb{Q}]\)

\[
C^\Delta_n(K) = \bigoplus_{n\text{-simplices}} \mathbb{Z}/2 = \text{free abelian group w/ basis } n\text{-simplices} \Rightarrow C^\Delta_n(K; F) = \bigoplus_{n\text{-simp}} F
\]

Define boundary maps just as before, get \( H^\Delta_n(K; F) = \ker \partial_n / \text{im } \partial_n \), a vector space over \( F \) w/ basis \( n\text{-simp} \).

Can do same for singular homology.

**Ex:** \( F = \mathbb{Z}/2 \). [In some sense, this is simpler than orig case]

\[
C^\Delta_n(K) = \bigoplus_{n\text{-simp}} \mathbb{Z}/2
\]

\( C \) is just a collection of \( n\text{-simplices} \)

\[
H_n(S^k; \mathbb{Z}/2) = \begin{cases} 
\mathbb{Z}/2 & n = 0, k \\
0 & \text{otherwise}
\end{cases}
\]

For \( P = \text{[proj plane]} \) we have

\[
\begin{array}{ccc}
\text{coeff} & \mathbb{Z} & \mathbb{Z}/2 \\
\hline
n \quad \text{mod } 2 & \begin{cases} 
0 & \text{if } n \neq 0, 2 \\
\mathbb{Z}/2 & \text{if } n = 0, 2
\end{cases}
\end{array}
\]

There are no orientations now:

\( \partial(C^{v_1} + C^{v_0}) = [v_0] + [v_1] \)

\( \partial(\triangle) = \triangle \)
Aside: S a cpt surface. Then
S contains a Möbius band $\Leftrightarrow H_2(S; \mathbb{Z}) = 0$

Proof: Suffices to show for any simp. complex

$\chi(K) = \sum (-1)^n \dim H_n(1K1; \mathbb{Z}/2)$

since the RHS is a top. invariant.

Consider the chain complex for $H_n^A(K; \mathbb{Z}/2)$:

$0 \rightarrow C_n(K; \mathbb{Z}/2) \rightarrow \ldots \rightarrow C_1(K; \mathbb{Z}/2) \rightarrow C_0(K; \mathbb{Z}/2) \rightarrow 0$

By HW we have

$\sum (-1)^n \dim C_n(K; \mathbb{Z}/2) = \sum (-1)^n \dim H_n^A(K; \mathbb{Z}/2)$

$\sum (-1)^n \dim C_n(K; \mathbb{Z}/2)$

$\chi(K)$

$\sum (-1)^n \dim H_n(K; \mathbb{Z}/2)$

as desired. \hfill \Box