Name:

- This is a closed-book, closed-notes exam. No electronic aids are allowed.

- Read each question carefully. Proof questions should be written out with all the details. You may use results proven in class, but you should explicitly cite the results being used.

- Answer the questions in the spaces provided on the question sheets. If you need extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

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1. **Section 1**

**Exercise 1.1.** (5 points) Find all the least nonnegative incongruent solutions of
\[12x \equiv 16 \pmod{32} \].

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**Exercise 1.2.** (5 points) Find the inverse modulo \( m = 81 \) of the integer \( n = 40 \) below
2. Section 2

Exercise 2.1. (10 points) State and prove Euler’s theorem. Make sure you quote any previous results clearly. If you use reduced residue systems, then make sure you state and prove any results you need.

Exercise 2.2. (5 points - bonus) How can one recover Fermat’s little theorem from Euler’s theorem?
Exercise 3.1. (10 points) Solve the following.

(a) What is the Euler totient function $\phi(n)$? Write $\phi(n)$ as a divisor sum.

(b) It is known that $\phi(n)$ is multiplicative. Disprove (via a counterexample) that $\phi(n)$ is completely multiplicative.

(c) How would you write $\phi(n)$ as a finite product over primes $p$ such that $p|n$? (You may state the formula from the course).

(d) Let $n \in \mathbb{N}$. If $p_1^{a_1}p_2^{a_2}\cdots p_m^{a_m}$ is the prime factorization of $n$, use part (c) above to prove that

$$\phi(n) = p_1^{a_1-1}p_2^{a_2-1}\cdots p_m^{a_m-1}\prod_{i=1}^{m}(p_i - 1).$$
4. Section 4

Exercise 4.1. (5 points) Use Fermat’s little theorem to find all incongruent solutions of

\[11x \equiv 15 \mod 29.\]

Exercise 4.2. (5 points) Use Euler’s theorem to find all incongruent solutions of

\[5x \equiv 21 \mod 36.\]
Exercise 4.3. (5 points) Use Wilson’s theorem to find the least nonnegative residue modulo $m = 67$ of $n = 64!$.

Exercise 4.4. (5 points) Find the least nonnegative solution of each system of congruences below.

\[
\begin{align*}
x \equiv 2 & \pmod{3} \\
x \equiv 1 & \pmod{5} \\
x \equiv 4 & \pmod{7}
\end{align*}
\]