1. Warm up

These questions are not to be turned in (review of the fundamental theorem of arithmetic). It is a valuable practice for the final.

**Exercise 1.1.** Find the greatest common divisor and the least common multiple of each pair of integers below.
   a) $2^2 \times 3^3 \times 5 \times 7$ and $2^2 \times 3^2 \times 5 \times 7^2$
   b) $2^2 \times 5^2 \times 7^3 \times 11^2$ and $3 \times 5 \times 11 \times 13 \times 17$

**Exercise 1.2.**

**Definition 1.1.** Let $n$ and $a$ be positive integers and let $p \in \mathbb{P}$. Then $p^a$ is said to exactly divide $n$, denoted $p^a | | n$, if $p^a$ and $p^{a+1} \not| n$.

Assume that $p^a | | m$ and $p^b | | n$.
   a) What power of $p$ exactly divides $m + n$? Prove your assertion.

**Exercise 1.3.** Solve the following.
   a) Let $n \in \mathbb{N}$ and let $p \in \mathbb{P}$. If $p|n!$, prove that the exponent of $p$ in the prime factorization of $n!$ is $[n/p] + [n/p^2] + [n/p^3] + \cdots$. Note that this sum if finite since $[n/p^m] = 0$ if $p^m > n$.
   b) Use part a) above to find the prime factorization of $20!$.
   c) Find the number of zeros with which the decimal representation of $100!$ terminates.

2. The real thing

These questions are due on Wednesday September 25th 2015.

**Exercise 2.1.** Let $a$ and $b$ be positive integers.
   a) Prove that $(a, b)[a, b]$.
   b) Find a prove a necessary and sufficient condition that $(a, b) = [a, b]$.

**Exercise 2.2.** Solve the following.
   a) Prove or disprove that $\{-39, 72, -23, 50, -15, 63, -52\}$ is a complete residue system modulo 7.
   b) Find a complete residue system modulo 7 consisting entirely of even integers.
   c) Find a complete residue system modulo 7 consisting entirely of odd integers.

**Exercise 2.3.** Solve the following.
   a) Let $a$ be an even integer. Prove that $a^2 \equiv 0 \mod 4$.
   b) Let $a$ be an odd integer. Prove that $a^2 \equiv 1 \mod 8$. Deduce that $a^2 \equiv 1 \mod 4$.
   c) Prove that if $n$ is a positive integer such that $n \equiv 3 \mod 4$, then $n$ cannot be written as the sum of two squares of integers.
   d) Prove or disprove the converse of the statement in part c) above.
Exercise 2.4. Recall what we talked about in class regarding divisibility by 9 (and hence 3). Prove that a positive integer $n$ is divisible by 11 if and only if the integer obtained by alternately adding and subtracting its digits (first add, then subtract, then add, then subtract etc...) beginning with adding the units and working to the left is divisible by 11. Example $n = 2728$, then $2 - 7 + 2 - 8 = -11$ and $11 | (-11)$ so 2728 is divisible by 11. On the other hand for $n = 31415$ we have $3 - 1 + 4 - 1 + 5 = 10$ and $11 / 10$, so 31415 is not divisible by 11.