Exercise 0.1. The condition is \( a = \pm b \). The sufficiency is straightforward. For the necessity, we need to recall the definitions of divisibility. Since \( a \mid b \) and \( b \mid a \), there exist integers \( c \) and \( d \) for which \( b = ac \) and \( a = bd \). If \( b = 0 \), then \( a = 0 \) as desired. If \( b \neq 0 \), then \( b = bdc \) so that \( 1 = dc \) which means that \( c = d = 1 \) or \( c = d = -1 \) and the desired result follows.

Exercise 0.2. The solutions are as follows.

a) A counterexample is given by \( a = c = 1, b = 2 \) and \( d = 3 \).

b) Since \( a \mid b \) and \( c \mid d \), there exist integers \( e \) and \( f \) for which \( b = ae \) and \( d = cf \). Then \( bd = (ac)(ef) \), and the desired result follows.

c) A counterexample is given by \( a = 2, b = 3 \) and \( c = 4 \).

Exercise 0.3. The solutions are as follows. However, this might not be the unique way to proceed.

a) We first need to establish the desired statement for all positive integers by induction. The desired statement holds for \( n = 1 \) clearly. Assuming that the statement holds for \( n = k \geq 1 \), we then have

\[
(k + 1)^3 - (k + 1) = (k^3 - k) + 3(k^2 + k).
\]

Since the right-hand side of this equation is divisible by 3, the induction is complete. Also note that the statement holds for \( n = 0 \). To establish the statement for all the negative integers, note that the first part of the proof implies that \( 3 \mid (n - n^3) \) for all positive integers \( n \), then use the trick that \( n - n^3 = (-n)^3 - (-n) \).

b) Again we use induction. The statement holds for \( n = 1 \). Assuming that it holds for \( n = k \geq 1 \), we then have

\[
(k + 1)^5 - (k + 1) = (k^5 - 5) + 5(k^4 + 2k^3 + 2k^2 + k).
\]

Since the right-hand side of this equation is divisible by 5, the induction is complete. Now, we do as we did before. The statement holds for \( n = 0 \), then note that the first part of the proof implies that \( 5 \mid (n - n^5) \) for all positive integers \( n \) and use the fact that \( n - n^5 = (-n)^5 - (-n) \).

c) A counterexample is given by \( n = 2 \).

Exercise 0.4. Let us disprove the conjecture by proving that the only prime number \( p \) for which \( p + 2 \) and \( p + 4 \) are also primes is \( p = 3 \). The cases where \( p = 2 \) and \( p = 3 \) are straightforward. Now let \( p \in \mathbb{P} \) with \( p > 3 \). Then either \( p + 1 \) or \( p + 2 \) is divisible by 3 since \( \{p, p + 1, p + 2\} \) are three consecutive positive integers. If \( p + 1 \) is divisible by 3, then so is \( p + 1 + 3 = p + 4 \), and hence \( p + 4 \) is not a prime. On the other hand, if \( p + 2 \) is divisible by 3, then \( p + 2 \) is not a prime. The result now follows.

Exercise 0.5. For this problem it helps to know the factorization

\[
a^{xy} - 1 = (a^x - 1)(a^{x(y-1)} + a^{x(y-2)} + \cdots + a^x + 1) = (a^x - 1) \sum_{k=1}^{y} a^{x(y-k)}.
\]

Let us try by contradiction (i.e. \( a \neq 2 \) and \( n \) is a composite number). The case \( a = 1 \) is straightforward. If \( a > 2 \), then the equation

\[
a^n - 1 = (a - 1)(a^{n-1} + a^{n-2} + \cdots + a + 1)
\]
provides us with a way to factorize $a^n - 1$. If $a \neq 2$, then $a^n - 1$ is not in $\mathbb{P}$. If $n$ is composite, e.g. $n = xy$, then we use a factorization of the type (0.1) to show that $a^n - 1$ is not a prime.

**Exercise 0.6.** Since $n^2 + 1$ is a prime number and $n \neq 1$, then $n$ must be even. So there exists an integer $m$ for which $n = 2m$ which yields $n^2 + 1 = (2m)^2 + 1 = 4(m^2) + 1$, as desired.