1. Warm up

The following are some warm up questions. Not to be turned in.

**Exercise 1.1.** Prove or disprove each statement below.
   a. 6|42
   b. 4|50
   c. 16|0
   d. 0|15
   e. 14|997157
   f. 17|998189

**Exercise 1.2.** Find integers \(a, b, c\) such that \(a|bc\) but \(a \not| b\) and \(a \not| c\).

**Exercise 1.3.** Find the unique integers \(q, r\) guaranteed by the division algorithm (Theorem 1.4) with each dividend and divisor below.
   a. \(a = 47, b = 6\)
   b. \(a = 281, b = 13\)
   c. \(a = 343, b = 49\)
   d. \(a = -105, b = 10\)
   e. \(a = -469, b = 31\)
   f. \(a = -500, b = 28\)

**Exercise 1.4.** Let \(a, b \in \mathbb{Z}\) with \(a|b\). Prove that \(a^n|b^n\) for every positive integer \(n\).

**Exercise 1.5.** Prove the following statements.
   a. Let \(n \in \mathbb{Z}\). Prove that \(n\) is an even integer if and only if (iff) \(n = 2m\) with \(m \in \mathbb{Z}\).
   b. Let \(n \in \mathbb{Z}\). Prove that \(n\) is an odd integer iff \(n = 2m + 1\) with \(m \in \mathbb{Z}\).
   c. Prove that the sum and product of two even integers are even.
   d. Prove that the sum of two odd integers is even and that their product is odd.
   e. Prove that the sum of an even integer and an odd integer is odd and that their product is even.

2. The real thing

These questions are due on Wednesday September 2nd 2015.

**Exercise 2.1.** If \(a, b \in \mathbb{Z}\), find a necessary and sufficient condition that \(a|b\) and \(b|a\).

**Exercise 2.2.** Prove or disprove the following statements.
   a. If \(a, b, c\) and \(d\) are integers such that \(a|b\) and \(c|d\), then \(a + c|b + d\).
   b. if \(a, b, c\) and \(d\) are integers such that \(a|b\) and \(c|d\), then \(ac|bd\).
   c. if \(a, b, c\) and \(d\) are integers such that \(a \not| b\) and \(b \not| c\), then \(a \not| c\).

**Exercise 2.3.** Solve the following.
   a. Let \(n \in \mathbb{Z}\). Prove that \(3|n^3 - n\).
b. Let \( n \in \mathbb{Z}. \) Prove that \( 5|n^5 - n. \)

c. Let \( n \in \mathbb{Z}. \) Is it true that \( 4|n^4 - n? \) Provide a proof or counterexample.

**Exercise 2.4.** Prove or disprove the following conjecture.

**Conjecture 2.1.** There are infinitely many prime numbers \( p \) for which \( p + 2 \) and \( p + 4 \) are also prime numbers.

**Exercise 2.5.** First we introduce the following definition.

**Definition 2.1.** Any prime number expressible in the form \( 2^p - 1 \) with \( p \) prime is said to be a Mersenne prime.

Let \( a \) and \( n \) be positive integers with \( n \neq 1. \) Prove that, if \( a^n - 1 \) is a prime number, then \( a = 2 \) and \( n \) is a prime number. Conclude that the only prime numbers of the form \( a^n - 1 \) with \( n \neq 1 \) are Mersenne primes.

**Exercise 2.6.** Let \( n \) be a positive integer with \( n \neq 1. \) Prove that, if \( n^2 + 1 \) is a prime number, then \( n^2 + 1 \) is expressible in the form \( 4k + 1 \) with \( k \in \mathbb{Z}. \)