Cos, Day 9

Last Time Euler sequences

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0 \quad \text{and its dual}$$

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

Lemma If $V = W \oplus W'$ as fin-dim'l k-vector spaces, then $\Lambda^n_k(V) \cong \bigoplus_{r=0}^{n} \Lambda^{n-r}(W) \otimes \Lambda^r(W')$.

Better Functoriality gives $\Lambda^n_k(W) \hookrightarrow \Lambda^n_k(V)$ whenever $W \subseteq V$.

Claim Quotient $\Lambda^n_k(V)/\Lambda^n_k(W) \cong \Lambda^{n-1}_k(W) \otimes V/W$.

Iterate Gives filtration

$$F_0 \subseteq F_1 \subseteq \dots \subseteq F_n \subseteq \Lambda^n_k(V)$$

with each $F_\ell / F_{\ell-1} \cong \Lambda^{n-\ell}_k(W) \otimes \Lambda^\ell_k(V/W)$.

Want to apply fiberwise to Euler sequence.
For each j, gives exact sequence

$$0 \to \underset{\substack{\cong \\ \Omega^j_{\mathbb{P}^n}}}{\Lambda^j \Omega^1_{\mathbb{P}^n}} \to \Lambda^j \mathcal{O}(-1)^{n+1} \to \underset{\substack{\cong \\ \Omega^{j-1}_{\mathbb{P}^n}}}{\Lambda^{j-1}\Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}} \to 0$$

$\underline{j=1}$: $0 \to H^0(\Omega^1) \to H^0(\cancel{\mathcal{O}(-1)^{n+1}}) \to H^0(\mathcal{O}) \simeq H^1(\Omega^1) \to \cancel{H^1(\mathcal{O}(-1)^{n+1})}$

$$\to \cancel{H^1(\mathcal{O})} \to H^2(\Omega^1) \to H^2(\mathcal{O}) \to \cdots$$

$$\Rightarrow \quad H^1(\Omega^1_{\mathbb{P}^n}) \cong k, \qquad H^i(\Omega^1_{\mathbb{P}^n}) = 0 \text{ for } i \neq 1.$$

Next, $\underline{j=2}$

$$0 \to H^0(\Omega^2) \to \cancel{H^0(\Lambda^2(\mathcal{O}(-1)^{n+1})) \to H^0(\Omega^1)}$$
$$\to H^1(\Omega^2) \to H^1(\Lambda^2(\mathcal{O}(-1)^{n+1})) \to H^1(\Omega^1)$$
$$\to H^2(\Omega^2) \to H^2(\Lambda^2(\mathcal{O}(-1)^{n+1})) \to H^2(\Omega^1) \to \cdots$$

<u>Rmk</u> $\Lambda^j(\mathcal{O}(-1)^{n+1}) = \mathcal{O}(-j)^{\binom{n+1}{j}}$ for $0 \leq j \leq n$.

So <u>all</u> $H^i(\Lambda^j(\mathcal{O}(-1)^{n+1})) = 0$, $0 \leq j \leq n$, $0 \leq i \leq n$, $(i,j) \neq (0,0)$.

In general get

$$H^{i-1}(\Omega^{j-1}) \to H^i(\Omega^j) \to H^i(\Lambda^j(\mathcal{O}(-1)^{n+1})) \to H^i(\Omega^{j-1}) \to \cdots$$

By induction on j, get

$$H^i(\mathbb{P}^n, \Omega^j) = \begin{cases} 0 & i \neq j \\ k & 0 \leq i = j \leq n. \end{cases}$$

<u>Hodge Numbers</u> of a variety X (projective, or at least proper) are

$$h^{i,j}(X) = \dim_k(H^j(X, \Omega^i)).$$

Aside

Thm Let $X \overset{i}{\subseteq} \mathbb{P}^n_k$ be a closed subscheme, $\mathcal{F}$ a coherent sheaf on X. Then
$$\dim_k H^i(X, \mathcal{F}) \text{ is finite for all } i \geq 0.$$

Pf sketch

(1) Use Čech complex to see that
$$H^i(X, \mathcal{F}) \cong H^i(\mathbb{P}^n, i_* \mathcal{F}) \text{ for all } i \geq 0.$$

(2) Use that every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is of the form $\mathcal{F} = \widetilde{M}$ for some $M \in k[x_0, \ldots, x_n]\text{-mod}_{gr}$ (f.g. graded modules).

(3) Use that every $M \in k[x_0 \ldots x_n]\text{-mod}_{gr}$ has a finite, graded-free resolution. [Hilbert Syzygy Theorem]

(4) Conclude that every $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}^n)$ has a finite resolution by ~~l~~ocally free sheaves $\bigoplus \mathcal{O}(n_j)$.

(5) Use an induction, plus corresponding fact for $H^i(\mathcal{O}(n_j))$, to prove it for all $\mathcal{F} \in \mathrm{Coh}(X)$.

Rmk Grothendieck proved w/ "projective" replaced by "proper", but proof very different.

(…), Day 10

$\boxed{\text{Ext}}$ For $(X, \mathcal{O}_X)$ a ringed space, $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$, define $\text{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, -)$ to be right derived functors of $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, -)$.

<u>Fact</u> If X is a locally noetherian scheme, then injective objects of $\text{Qcoh}(X)$ are the same as injective objects of $\mathcal{O}_X\text{-Mod}$ that are quasicoherent as sheaves.

Pf. Hartshorne, Residues and Duality, Thm 7.18. $\square$.

<u>Fact</u> If X is a scheme, $\text{Qcoh}(X)$ has enough injectives.

Pf. Enochs & Estrada, 2005.

<u>Conclusion</u> The two types of $\text{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, -)$ for $\mathcal{F} \in \text{Qcoh}(X)$, computed in either $\text{Qcoh}(X)$, or $\mathcal{O}_X\text{-Mod}$, agree for $\mathcal{G} \in \text{Qcoh}(X)$.

<u>Prop</u> If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence in $\mathcal{O}_X\text{-Mod}$, get a long exact sequence

$$\text{Ext}^i_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \to \text{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \text{Ext}^i_{\mathcal{O}_X}(\mathcal{F}', \mathcal{G}) \to \text{Ext}^{i+1}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \to \cdots$$

Pf. Use injective resolution $\mathcal{G} \to I^\bullet$ and compute complexes. $\square$.

<u>Thus</u> If $L_\bullet \to \mathcal{F}$ is a locally free resolution of $\mathcal{F}$, can in principle compute $Ext^i_X(\mathcal{F},\mathcal{G})$ in terms of $Ext^j_X(L_\ell, \mathcal{F})$ for the various j, ℓ.

<u>Prop</u> Let L be locally free of finite rank, $L^\vee = \underline{Hom}_{\mathcal{O}_X}(L, \mathcal{O}_X)$ its dual. Then for any $\mathcal{F}, \mathcal{G} \in \mathcal{O}_X\text{-Mod}$,

$$Ext^i_X(\mathcal{F}\otimes L, \mathcal{G}) \cong Ext^i_X(\mathcal{F}, L^\vee \otimes \mathcal{G}).$$

Pf. For $i=0$, clear.(?) Then both sides are δ-functors in $\mathcal{G}$ for fixed $\mathcal{F}, L$, both vanish for $i>0$ when $\mathcal{G}$ is injective, follows from general nonsense. $\square$.

<u>Serre Duality, Part I</u> $X = \mathbb{P}^n_k$, k a field. Then

(i) $H^n(\mathbb{P}^n, \Omega^n) \cong k$. Fix such an isomorphism.

(ii) For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$, get perfect pairing

$$Ext^i(\mathcal{F}, \Omega^n_{\mathbb{P}^n}) \times Ext^{n-i}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{F}) \to H^n(\mathbb{P}^n, \Omega^n) \xrightarrow{\sim} k$$

for all $0 \le i \le n$.

It's functorial in $\mathcal{F}$, as well.

<u>First</u> show (ii) for $i=0$, i.e.

$$\begin{array}{c}\mathrm{Ext}^0(\mathcal{F},\Omega^n) \times H^n(\mathcal{F}) \longrightarrow H^n(\mathbb{P}^n,\Omega^n) \cong k \\ \| \\ \mathrm{Hom}(\mathcal{F},\Omega^n)\end{array}$$

perfect. Clear what pairing should be:

$$\langle u,\varphi\rangle = H^n(u)(\varphi) \in H^n(\mathbb{P}^n,\Omega^n).$$

Clearly nondegenerate for $\mathcal{F} = \bigoplus_{i=1}^{m} \mathcal{O}(d_i)$.

For arbitrary $\mathcal{F}$, write $\mathcal{F} \cong \mathrm{coker}\left(\bigoplus_{i=1}^{m} \mathcal{O}(d_i) \to \bigoplus_{j=1}^{\ell} \mathcal{O}(e_j)\right)$

Get exact sequences $H^n\left(\bigoplus_{i=1}^{m}\mathcal{O}(d_i)\right) \to H^n\left(\bigoplus_{j=1}^{\ell}\mathcal{O}(e_j)\right) \to H^n(\mathcal{F}) \to 0$,

$$0 \to \mathrm{Hom}(\mathcal{F},\Omega^n) \to \mathrm{Hom}\left(\bigoplus_{j=1}^{\ell}\mathcal{O}(e_j),\Omega^n\right) \to \mathrm{Hom}\left(\bigoplus_{i=1}^{m}\mathcal{O}(d_i),\Omega^n\right)$$

Dualizing second and using pairings, get

$$\begin{array}{ccccccc} H^n\left(\bigoplus_{i=1}^{m}\mathcal{O}(d_i)\right) & \to & H^n\left(\bigoplus_{j=1}^{\ell}\mathcal{O}(e_j)\right) & \to & H^n(\mathcal{F}) & \to & 0 \\ \wr\wr & & \wr\wr & & \searrow & & \\ \mathrm{Hom}\left(\bigoplus\mathcal{O}(d_i),\Omega^n\right)^* & \to & \mathrm{Hom}\left(\bigoplus\mathcal{O}(e_j),\Omega^n\right)^* & \to & \mathrm{Hom}(\mathcal{F},\Omega^n)^* & \to & 0 \end{array}$$

get $H^n(\mathcal{F}) \cong \mathrm{Hom}(\mathcal{F},\Omega^n)^*$ as desired.

Now, show (ii) for all i. Use that each of

$H^{n-i}(\mathcal{F})^*$, $\mathrm{Ext}^i(\mathcal{F},\Omega^n)$ is a contravariant δ-functor in $\mathcal{F}$ with same $i=0$ term. By a result of Grothendieck, enough to show both are coeffaceable, ie. for any $\mathcal{F}$, exists $\mathcal{G} \overset{u}{\twoheadrightarrow} \mathcal{F}$ with u inducing trivial (zero) map

$$\mathrm{Ext}^i(\mathcal{F}, \Omega^n) \xrightarrow{\mathrm{Ext}^i(u,1)} \mathrm{Ext}^i(\mathcal{G}, \Omega^n)$$

for $i>0$.

For any $\mathcal{F}$, find $\bigoplus_{j=1}^{m} \mathcal{O}(d_j) \twoheadrightarrow \mathcal{F}$

with $d_i \ll 0$. Then

$$\mathrm{Ext}^i(\bigoplus \mathcal{O}(d_j), \Omega^n) = \bigoplus \mathrm{Ext}^i(\mathcal{O}(d_j), \Omega^n)$$

$$= \bigoplus H^i(\Omega^n(d_j)).$$

Recall $\Omega^n \cong \mathcal{O}(-n-1)$. So these all vanish for $i>0$.

Similarly need $\mathcal{G} \overset{u}{\twoheadrightarrow} \mathcal{F}$ with

$$H^{n-i}(u)^* : H^{n-i}(\mathcal{F})^* \longrightarrow H^{n-i}(\mathcal{G})^*$$

zero for $i>0$. Clear that same $\mathcal{G}$ will do! $\square$

What about Serre Duality for other closed $X \overset{i}{\subseteq} \mathbb{P}^n$?

Have

$$\mathrm{Ext}^{n-i}_{\mathbb{P}^n}(i_*\mathcal{F}, \Omega^n_{\mathbb{P}^n}) \xrightarrow{\sim} H^i(\mathbb{P}^n, i_*\mathcal{F})^*$$

$$\cong H^i(X, \mathcal{F})^*$$

Suppose i_* has a right adjoint! Then we should, writing $i^!$ for it, get

$$\text{LHS} \cong \mathrm{Ext}^{n-i}_X(\mathcal{F}, i^!\Omega^n_{\mathbb{P}^n}).$$

<u>Fact</u> $i^!$ <u>does</u> exist, as long as we are willing to work with derived categories, and

$$i^! \Omega^n_{\mathbb{P}^n} \cong \Omega^n_{\mathbb{P}^n} \otimes \Lambda^d (I_X/I_X^2)^\vee \, [-d]$$

where $d = \operatorname{codim}(X \subseteq \mathbb{P}^n)$ and

$$\Lambda^d (I_X/I_X^2)^\vee = \operatorname{Hom}_{\mathcal{O}_X}(\Lambda^d_{\mathcal{O}_X}(I_X/I_X^2), \mathcal{O}_X),$$

<u>provided</u> X is, for example, a local complete intersection. For example if X is smooth,

$$i^! \Omega^n_{\mathbb{P}^n} = \Omega^{n-d}_X [-d].$$

Then Serre Duality on X becomes

$$\operatorname{Ext}^{D-i}_X(\mathcal{F}, \Omega^D_X) \cong H^i(X, \mathcal{F})^* \text{ if } D = \dim X.$$

More generally,

$$i^! \Omega^n_{\mathbb{P}^n} = \mathcal{E}xt^d_{\mathbb{P}^n}(\mathcal{O}_X, \Omega^n_{\mathbb{P}^n})[-d]$$

for $d = \operatorname{codim}(X \subseteq \mathbb{P}^n)$.

Computing this, or perhaps the <u>dualizing</u> <u>sheaf</u>

$$\omega_X = \mathcal{E}xt^d_{\mathbb{P}^n}(\mathcal{O}_X, \Omega^n_{\mathbb{P}^n}),$$

is challenging in general. Even after decades, there are things not well understood.