Last Time Euler sequences

\[ 0 \to \Omega^1_{p^n} \to (\mathcal{O}(-1)^{n+1}) \to T_{p^n} \to 0 \quad \text{and its dual} \]

\[ 0 \to \Omega^1_{p^n} \to (\mathcal{O}(-1)^{n+1}) \to (\mathcal{O}_{p^n})^\vee \to 0. \]

Lemma If \( V = W \oplus W' \) as \( k \)-vector spaces, then

\[ \bigwedge_k^r(V) \cong \bigoplus_{r=0}^{\infty} \bigwedge_k^r(W) \otimes \bigwedge_k^r(W'). \]

Better functoriality gives \( \bigwedge_k^r(W) \to \bigwedge_k^r(V) \) whenever \( W \subseteq V. \)

Claim Quotient \( \bigwedge_k^n(N)/\bigwedge_k^n(W) \cong \bigwedge_k^{n-1}(W) \otimes \bigwedge_k^1(N/W). \)

Iterate Gives filtration

\[ F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n \subseteq \bigwedge_k^n(N) \]

with each \( F_j/F_{j-1} \cong \bigwedge_k^{n-j}(W) \otimes \bigwedge_k^j(N/W). \)

Want to apply otherwise to Euler sequence.

For each \( j \), gives exact sequence

\[ 0 \to \bigwedge_j \bigwedge^1_{p^n} \to \bigwedge_j \mathcal{O}(-1)^{n+1} \to \bigwedge^{j-1} \bigwedge^1_{p^n} \otimes \mathcal{O}_{p^n} \to 0 \]

\[ \bigwedge_j \bigwedge^1_{p^n} \]

\[ \bigwedge^{j-1} \bigwedge^1_{p^n} \]
\[ j \geq 0 \Rightarrow H^0(\Omega^{-1}) \rightarrow H^0(\Omega^{-1})^{n+1} \rightarrow H^0(\Omega) \cong H^0(\Omega^1) \rightarrow H^1(\Omega) \rightarrow H^2(\Omega^1) \rightarrow \cdots \]

\[ \Rightarrow H^i(\Omega^1)^{m \times n} = k, \quad H^i(\Omega^1)^{m \times n} = 0 \text{ for } i \neq 1. \]

Next, \( j \geq 2 \)

\[ 0 \rightarrow H^0(\Omega) \rightarrow H^0(\Lambda^2(\Omega^{-1})^{n+1}) \rightarrow H^0(\Omega^1) \]

\[ \rightarrow H^1(\Omega) \rightarrow H^1(\Lambda^2(\Omega^{-1})^{n+1}) \rightarrow H^1(\Omega^1) \]

\[ \rightarrow H^2(\Omega) \rightarrow H^2(\Lambda^2(\Omega^{-1})^{n+1}) \rightarrow H^2(\Omega^1) \rightarrow \cdots \]

\[ \text{rank } \Lambda^j(\Omega^{-1})^{n+1} = (n-j)^{n+1} \text{ for } 0 \leq j \leq n. \]

So all \( H^j(\Lambda^j(\Omega^{-1})^{n+1}) = 0, \quad 0 \leq j \leq n, \quad (n-j) \neq (0, 0). \)

In general get

\[ H^i(\Omega^1) \rightarrow H^i(\Omega^1) \rightarrow H^i(\Lambda^j(\Omega^{-1})^{n+1}) \rightarrow H^i(\Omega^{j-1}) \rightarrow \cdots \]

By induction on \( j \), get

\[ H^i(\Omega^1) = \begin{cases} 0 & i \neq j^* \\ k & 0 \leq i = j \leq n \end{cases} \]

Hodge Numbers of a variety \( X \) (projective, or at least proper) are

\[ h^{ij}(X) = \text{dim}_k (H^i(X, \Omega^j)). \]
Aside

Let $X \subset \mathbb{P}^n$ be a closed subscheme, of a coherent sheaf on $X$. Then $\dim H^i(X, \mathcal{F})$ is finite for all $i \geq 0$.

**Proof sketch**

1. Use Čech complex to see that $H^i(X, \mathcal{F}) \cong H^i(\mathbb{P}^n, (\mathcal{F}_x))$ for all $i \geq 0$.

2. Use that every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is of the form $\mathcal{F} = \mathcal{M}$ for some $\mathcal{M} \in \mathcal{K} \oplus \cdots \oplus \mathcal{K}$-modules.

3. Use that every $\mathcal{M} \in \mathcal{K} \oplus \cdots \oplus \mathcal{K}$-modules has a finite graded-free resolution, [Hilbert Syzygy Theorem].

4. Conclude that every $\mathcal{F} \in \text{Coh}(\mathbb{P}^n)$ has a finite resolution by locally free sheaves $\bigoplus \mathcal{O}(n_\mathcal{G})$.

5. Use an induction, plus corresponding fact for $H^i(\mathcal{O}(n_\mathcal{G}))$, to prove it for all $\mathcal{F} \in \text{Coh}(X)$.

**Remark** Grotendieck proved W/ "projective" replaced by "proper", but proof very different.
For $(X, O_X)$ a ringed space, $F \in O_X$-$\text{Mod}$, define $\text{Ext}^i_{O_X}(F, -)$ to be right derived functors of $\text{Hom}_{O_X}(F, -)$.

**Fact.** If $X$ is a locally noetherian scheme, then injective objects of $\text{Qcoh}(X)$ are the same as injective objects of $O_X$-$\text{Mod}$ that are quasi-coherent as sheaves.

*Pt. Hartshorne, Residues and Duality, Thu 7.16.*

**Fact.** If $X$ is a scheme, $\text{Qcoh}(X)$ has enough injectives.

*Pt. Enochs & Estrada, 2005.*

**Conclusion.** The two types of $\text{Ext}^i_{O_X}(F, -)$ for $F \in \text{Qcoh}(X)$, computed in either $\text{Qcoh}(X)$ or $O_X$-$\text{Mod}$, agree for $G \in \text{Qcoh}(X)$.

**Remark.** If $0 \to F' \to F \to F'' \to 0$ is a short exact sequence in $O_X$-$\text{Mod}$, get a long exact sequence

$$\text{Ext}^i_{O_X}(F'', G) \to \text{Ext}^i_{O_X}(F, G) \to \text{Ext}^i_{O_X}(F', G) \to \text{Ext}^{i+1}_{O_X}(F'', G) \to \cdots$$
Pt. Use injective resolution \( G \rightarrow I \) and compute completions.

Thus if \( L \rightarrow G \) is a locally free resolution of \( F \), can in principle compute \( \text{Ext}^i(L, F) \) in terms of \( \text{Ext}^j(L, G) \) for the various \( j, i \).

Prop. Let \( E \) be locally free of finite rank, \( L = \text{Hom}_X(L, 0_X) \) its dual. Then for any \( F, G \in 0_X - \text{Mod} \),

\[
\text{Ext}^i(\text{Hom}_X(L, G), F) \cong \text{Ext}^i(F, L \otimes F).
\]

Pt. For \( i = 0 \), clear (?) Then both sides are \( S \)-homologous in \( G \) for fixed \( E, L \), both vanish for \( i > 0 \) when \( G \) is injective, follows from general nonsense.

Serre Duality, Part 1 \( X = \mathbb{P}^n_k \) Then

(i) \( H^n(\mathbb{P}^n_k, \Omega^n) \cong k \). Fix such an isomorphism.

(ii) For any coherent sheaf \( F \) on \( \mathbb{P}^n_k \), get perfect pairing

\[
\text{Ext}^i(\Omega^n, F) \times \text{Ext}^n-i(0^n, F) \rightarrow H^n(\mathbb{P}^n, 0^n) \cong k
\]

for all \( 0 \leq i \leq n \).

It's functorial in \( F \), as well.
First show (ii) for i=0, i.e.,
\[ \text{Ext}^0(\mathcal{F}, \mathcal{O}^n) \times H^n(T) \to \mathcal{H}^n(\mathcal{F}, \mathcal{O}^n) \cong k \]
\[ \text{Hom}(\mathcal{F}, \mathcal{O}^n) \]
perfect. Clear what pairing should be:
\[ \langle u, \varphi \rangle = H^n(u)(y) \in H^n(\mathcal{F}, \mathcal{O}^n). \]
Clearly nondegenerate for \( \mathcal{F} = \bigoplus_{i=1}^r \mathcal{O}(d_i) \).

For arbitrary \( \mathcal{F} \), write \( \mathcal{F} \cong \text{colim} \bigoplus_{i=1}^r \mathcal{O}(d_i) \rightarrow \bigoplus_{j=1}^s \mathcal{O}(e_j) \)
get exact sequences
\[ H^n(\bigoplus_{i=1}^r \mathcal{O}(d_i)) \rightarrow H^n(\bigoplus_{j=1}^s \mathcal{O}(e_j)) \rightarrow H^n(\mathcal{F}) \rightarrow 0, \]
\[ 0 \rightarrow \text{Hom}(\mathcal{F}, \mathcal{O}^n) \rightarrow \text{Hom}(\bigoplus_{j=1}^s \mathcal{O}(e_j), \mathcal{O}^n) \rightarrow \text{Hom}(\bigoplus_{i=1}^r \mathcal{O}(d_i), \mathcal{O}^n) \]

Dualizing second and using pairings, get
\[ H^n(\bigoplus_{i=1}^r \mathcal{O}(d_i)) \rightarrow H^n(\bigoplus_{j=1}^s \mathcal{O}(e_j)) \rightarrow H^n(\mathcal{F}) \rightarrow 0 \]
\[ \text{Hom}(\bigoplus_{i=1}^r \mathcal{O}(d_i), \mathcal{O}^n) \rightarrow \text{Hom}(\bigoplus_{j=1}^s \mathcal{O}(e_j), \mathcal{O}^n) \rightarrow \text{Hom}(\mathcal{F}, \mathcal{O}^n) \]
get \( H^n(\mathcal{F}) \cong \text{Hom}(\mathcal{F}, \mathcal{O}^n)^* \) as desired.

Now, show (iii) for all i. Use that each of
\[ H^n(\mathcal{F}), \text{ Ext}^i(\mathcal{F}, \mathcal{O}^n) \]
is a contravariant \( \mathcal{O} \)-functor with same i=0 term. By a result of Grothendieck, enough to show both are
coefficientable, i.e., for any \( f \), exists \( g \rightarrow f \) with

\[
\text{Ext}^i(f, \Omega^n) \xrightarrow{\text{Ext}^i(\text{id}, 1)} \text{Ext}^i(g, \Omega^n)
\]

for \( i > 0 \).

For any \( f \), and \( \otimes \Omega(\epsilon_f) \rightarrow g \)

with \( d \neq 0 \). Then

\[
\text{Ext}^i(\otimes \Omega(\epsilon_f), \Omega^n) = \oplus \text{Ext}^i(\Omega(\epsilon_f), \Omega^n)
\]

Recall \( \Omega^n \cong \Omega(-n-1) \). So these all vanish for \( i > 0 \).

Similarly need \( g \rightarrow f \) with

\[
H^{n-i}(\text{cycl}) : H^{n-i}(f) \xrightarrow{\phi^{-1}} H^{n-i}(g)
\]

zero for \( i > 0 \). Clear that same \( g \)

will do!

What about Serre Duality for other closed \( X \subseteq \mathbb{P}^n \)?

Have

\[
\text{Ext}_{\mathbb{P}^n}^i(c_\ast F, \Omega^m) \cong H^{m-i}((\mathbb{P}^n, c_\ast F)\times X)
\]

Suppose \( c_\ast \) has a right adjoint! Then we should,

writing \( c_\ast \) for it, get

\[
LHS \cong \text{Ext}_{\mathbb{P}^n}^m(F, c_\ast \Omega^m).
\]
Fact 1 does exist, as long as we are willing to work with derived categories, and
\[ i^! \Omega^n \cong \Omega^n_{\mathcal{P}^n} \otimes \Lambda^d(I_x/I^d_x)[-d] \]
where \( d = \text{codim} (X \subseteq \mathcal{P}^n) \) and
\[ \Lambda^d(I_x/I^d_x) = \text{Hom}_{\mathcal{P}^n}(\Lambda^d/I^d_x, \Omega^d_{\mathcal{P}^n}) \]
provided \( X \) is, for example, a local complete intersection. For example, if \( X \) is smooth,
\[ i^! \Omega^n_{\mathcal{P}^n} = \Omega^n_X [-d] \]
then Serre Duality on \( X \) becomes
\[ \text{Ext}^{d-1}_X(\mathcal{F}, \Omega^d_X) \cong H^1(X, \mathcal{F}^*) \] if \( D = \dim X \).
More generally,
\[ i^! \Omega^n_{\mathcal{P}^n} = \otimes \text{Ext}^d_{\mathcal{P}^n}(\Omega^n_{\mathcal{P}^n}, \Omega^n_{\mathcal{P}^n})[-d] \]
for \( d = \text{codim} (X \subseteq \mathcal{P}^n) \).
Computing this, or perhaps the dualizing sheaf
\[ \omega_X = \text{Ext}^d_{\mathcal{P}^n}(\Omega^n_{\mathcal{P}^n}, \Omega^n_{\mathcal{P}^n}) \]
is challenging in general. Even after decades, there are things not well understood.