**X** top space, \( U \subseteq X \) opens.

\( \text{Top}(X) : \) category with open sets as objects, inclusions as morphisms.

Presheaf valued in category \( \mathcal{C} \): functor \( \text{Top}(X) \to \mathcal{C} \),

\[ \mathcal{F} : U \mapsto \mathcal{F}(U) \]

Most common \( \mathcal{C} = \text{Ab} \) category of abelian groups

\( \mathcal{F}(\emptyset) \) is a \( \mathcal{C} \) object

\[ \mathcal{F}(\emptyset) \]

**Ex.** Constant presheaf

\[ \text{Fun}(X, S) \]

Sheaf presheaf sat. for all \( U \subseteq X \) open, \( U = \bigcup_{i \in I} U_i \) open,

\[ \ast \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{(i, j) \in IXI} \mathcal{F}(U_i \cap U_j) \]

is exact. [unpick?]

**Ex.** Constant presheaf not a sheaf.

- "Constant sheaf" = locally constant functions
- Functions with "behavior prescribed locally"
- Sections of a map \( Y \to X \).

Stalk of \( \mathcal{F} \) at \( p \in X \) : \( \lim_{\to \mathcal{F}(U)} \)

Local family : \( \{(V, s) | p \in V \subseteq U, s \in \mathcal{F}(V)\} \)

\((V, s) \sim (V', s') \) if \( \exists p \in V \cap V' \) with \( s|_{V' \cap V} = s'|_{V' \cap V} \)
Generally stacks are unwieldy gadgets:

Ex. $X = \mathbb{A}^n_C$ complex affine $n$-space, $\mathcal{F} = \mathcal{O}_X$ sheaf of regular functions on $\mathbb{A}^n_C$. Do you know what that means? Then

$$\mathcal{U}_X(a_1, \ldots, a_n) = \mathbb{C}[x_1, \ldots, x_n]/(x_1-a_1, \ldots, x_n-a_n).$$

Huge ring.

Def. $\varphi: \mathcal{F} \to \mathcal{G}$ homomorphism of sheaves of abelian groups. [Meaning?] $\ker(\varphi)(U) = \ker(\varphi(U))$. It's a sheaf.

$\operatorname{im}(\varphi)(U) = \operatorname{im}(\varphi(U))$. It's only a presheaf in general.

SHEAFIFICATION $\mathcal{F} \to \mathcal{F}^+$. Read about it for Thursday?

Def. $\mathcal{F}^+ = (\operatorname{im}(\varphi))^+$.

Ex. $\mathcal{O}^\text{an}_C$ holo. hms on C. $(\mathcal{O}^\text{an}_C)^x$ nonvanishing holo. hms.

Claim. $\mathcal{F}^+ = (\mathcal{O}^\text{an}_C \xrightarrow{\exp} (\mathcal{O}^\text{an}_C)^x)$ is not a sheaf.

Work on 1.1–1.8 in §1.1 together.
Sheafification of a presheaf on top space $X$.

Functions $\mathcal{F}^+ : \text{AbShv}(X) \to \text{AbShv}(X) : \text{Forget}$,

$\mathcal{F}^+(U) = \{ \text{sections of } \mathcal{F} \mid \text{on } P \in \mathcal{U} \}$

$\mathcal{F}^+(U) = \{ \text{sections of } \mathcal{F} \mid \text{on } P \in \mathcal{U} \}$

with $s(p) = \text{image of } s$ and $s \in \mathcal{F}(U)$ in $\mathcal{F}^+$.

Image $\gamma : \mathcal{F} \to \mathcal{G}$.

\[ \text{Im}(\mathcal{F})(U) = \text{Im}(\mathcal{G})(U) \to \mathcal{G}(U). \]

\[ \text{Im}(\mathcal{F}) = \text{Im}(\gamma). \]

Exact sequence of sheaves

$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$

with $\ker(\gamma) = \text{im}(\mathcal{F})$, $\ker(\mathcal{H}) = 0$, $\text{im}(\gamma) = \mathcal{H}$.

Example $X = \mathbb{C}$ with Euclidean topology, or any nonsingular quasi-projective (say) complex variety with analytic topology and sheaf of holomorphic functions.

Claim $0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$ is exact.
Quasicoherent sheaves on a variety or scheme $X$.

Let $X$ be a variety or scheme, $O_X$ its sheaf of regular functions or structure sheaf.

So e.g. recall if $U \subseteq X$ and $U \subseteq \mathbb{A}^n$

closed affine subvariety,

$O_X(U) = \{ f : U \to k \mid f \in k[x_1, \ldots, x_n] \text{ with} \}

A sheaf of $O_X$-modules $\mathcal{F}$ is a sheaf of abelian groups on $X$ (topology is a given) with maps $O_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U) \quad \forall \ U \in \text{Top}(X)$

making each $\mathcal{F}(U)$ an $O_X(U)$-module, and so that for all $V \subseteq U \subseteq X$, the square

$O_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$

$\mathcal{F}(V) \downarrow \quad \downarrow$

$O_X(U) \times \mathcal{F}(V) \to \mathcal{F}(V)$

They form a category

An $O_X$-module is quasicoherent if for every $p \in X$

$\exists$ open nbhd $p \in U \subseteq X$ with exact sequence

$0 \to O_X(U) \to \mathcal{F}(U) \to \mathcal{F}_U \to 0$

of $O_X$-modules.

An $O_X$-module is (locally) finitely generated if

$\forall p \in X \exists p \in U \subseteq X$ and surjection $O_X(U) \to \mathcal{F}_U$.
An $O_X$-module $F$ is coherent if it is finitely generated and for all open sets $U \subseteq X$ and surjections $\phi^U : \mathcal{O}_X^n \to \mathcal{F}|U$, $\ker(\phi^U)$ is finitely generated.

Examples
- Ideal sheaves of closed subvarieties/subschemes.
- Structure sheaves of closed subvarieties/subschemes.
- $(\mathcal{O}_X^m)$ for $m \in \mathbb{Z}$ — do you know this one??
Grothendieck Topologies: Zariski topology is very coarse.

E.g., if $k$ is alg. closed field, $X, Y$ two irreducible curves over $k$, then $\#X = \#Y$ (cardinalities). Choose any bijection $X \sim Y$.

Claim: It's a Zariski homeomorphism.

Pf. Closed subsets of a curve are:

- entire curve
- finite sets
- empty set

Grothendieck topologies: more flexible notion of "topology" to get more open sets.

Def. A topology/site $T$ is a category $\text{cat}(T)$ with subclass $\mathcal{O}$ of collections of morphisms $\text{cov}(T) \ni \{U_i \to U\}_{i \in I}$, $U_i, U \in \mathcal{O}(\text{cat}(T))$ satisfying:

(T1) If $\{U_i \to U\}_{i \in I} \in \text{cov}(T)$ and $V \to U$ is morph. in $\text{cat}(T)$, then fiber products $V_i \times_U V$ exist and $\{V_i \times_U V \to V\}_{i \in I} \in \text{cov}(T)$.
(72) If \( \{ U_i \to V \}_{i \in I} \), \( \{ V_j \to U_i \}_{j \in J} \) are coverings in \( \text{cov}(\mathcal{T}) \), so is
\[
\{ V_j \to U \}
\]
\( \bigcup_{i \in I} U_i \) = \( \bigcup_{j \in J} V_j \).

(73) If \( U' \subseteq U \) is an isomorphism in \( \text{cat}(\mathcal{T}) \), then \( \text{cov}(U) \) lies in \( \text{cov}(U') \).

**Exercise (2) Zariski open coverings:**

(1) \( X \) a (locally path-connected Hausdorff) top space, \( \mathcal{T} \) a topology on \( X \), \( \mathcal{C} \) a cover of \( X \) by sets \( U_i \to X \), \( \mathcal{C}_j \) a covering of \( U \) where each \( U_i \to U \) is a covering space of an open subset of \( U \), so that a union of images of \( U \) is \( U \).

**Main Point**

(1) composition of covering maps is a covering \( \mathcal{C} \) map,
(2) fiber product with a covering is a covering.

*Zariski topology, Etale topology.*
Étale morphisms

$f: X \to Y$ schemes is locally finitely presented if $U \subset X$ affine open with $x \in U \cap f^{-1}(U) \subset Y$, $f(U) \subset V$, so $(\mathcal{O}_Y(V) \to \mathcal{O}_X(U))$ is finitely presented. [OK?]

It is flat if $V \subset X$, $(\mathcal{O}_Y(V) \to \mathcal{O}_X(U))$ is flat. [OK?]

$f: X \to Y$ locally presented is étale if it is flat and unramified, where unramified means $V \subset Y$, $X \times_Y \text{Spec}(k_j) = \bigsqcup_{j \neq 0} \text{Spec}(k_j)$, where each $(k_j) \to k_j$ is a finite separable field extension.

Example: $X, Y$ over $k$ of char $0$, then unramified means fibers are disjoint union of Spec of fields.

So, $A_k^1 \to \mathbf{A}_k^1$ not unramified over $t=0$, $\text{Spec}(k[T]/(T^2))$.

Example: Open immersions are étale. Composites, fiber products of étale morphisms are étale.

Étale site: $X_{\text{ét}}$: objects are étale $X$-schemes, coverings are families with unram. images equal
Artin Comparison Theorem: Let $X$ be a smooth $\mathbb{C}$-scheme of finite type, $X^an$ its associated analytic space, $F$ a locally constant sheaf of finite abelian groups on $X_{et}$, induces one on $X^an$ as well, and $\bar{H}^i(X_{et}, F) \cong H^i(X^an, F^an)$. You don't know what a Serre sheaf is yet. I suppose... We'll get there.
Recall for sheaves on any top. space \( X \), the global section functor \( \Gamma : \text{AbShv}(X) \to \text{Ab} \) is defined by
\[
\Gamma(F) = \Gamma(X, F) := \mathcal{F}(X).
\]
If \( X \) is a scheme, define \( \Gamma : \text{Qcoh}(X) \to \text{P}(X, \mathcal{O}_X) - \text{Mod} \) so that
\[
\text{Qcoh}(X) \xrightarrow{\Gamma} \text{P}(X, \mathcal{O}_X) - \text{Mod} \downarrow \Rightarrow \text{AbShv}(X) \xrightarrow{\Gamma} \text{Ab}
\]
commutes.

If \( X = \text{Spec}(R) \), then \( \text{P}(X, \mathcal{O}_X) = R \).

Thm [H, Cor 5.5.5]: If \( X = \text{Spec}(k) \), it always have adjoint pair of functors
\[
(\cdot)^*: \text{R-Mod} \to \text{Qcoh}(	ext{Spec}(R)): \Gamma
\]
which are equivalences of abelian categories.

In particular, \( \Gamma \) is exact, i.e., if
\[
0 \to F_1 \to F_2 \to F_3 \to 0
\]
is a short exact sequence in \( \text{Qcoh}(	ext{Spec}(R)) \), then
\[
0 \to \Gamma(F_1) \to \Gamma(F_2) \to \Gamma(F_3) \to 0
\]
is exact.

Fails for general schemes, e.g.,
\[
\begin{align*}
\mathcal{O}_{\mathbb{P}^1}^2 & \to \mathcal{O}_{\mathbb{P}^1(2)} \\
(x^2, z^2) & \to (x^2, x^2) & \text{surjective.}
\end{align*}
\]
Cohomology: Designed to measure failure of exactness of $F$.

**General Setup:**
- $C, D$: Abelian categories.
- (5) Initial = final object.
- (6) Hom-sets are abelian groups (i.e., endowed with that structure); composition laws are bilinear;
- (7) Finite direct sums exist in $C, D$;
- (8) every morphism has kernel, cokernel (universal properties);
- (9) every monomorphism is kernel of its cokernel;
- (10) every epimorphism is cokernel of its kernel;
- (11) every morphism factors as epimorphism followed by monomorphism.

**Additive Functor $F: C \to D$ is left exact if**

For all exact sequences $0 \to M \to N \to P$ in $C$,

$$F(0) \to F(M) \to F(N) \to F(P)$$

is exact.

Cohomology: additional collection of functors meant to "measure" failure of $F$ to be right exact.
Def: A $D$-functor from $C$ to $D$ is a collection of functors $T = (T^i)_{i \geq 0}$ together with morphisms $S^i: T^i(A'') \rightarrow T^{i+1}(A')$ for each s.e.s. $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ in $C$, $i \geq 0$, satisfying:

1) for each s.e.s. $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ in $C$,

$$0 \rightarrow T^0(A') \rightarrow T^0(A) \rightarrow T^0(A'') \xrightarrow{S^0} T^1(A') \rightarrow \cdots$$

$$\rightarrow T^i(A'') \rightarrow T^i(A) \rightarrow T^i(A'') \xrightarrow{S^i} T^{i+1}(A') \rightarrow \cdots$$

is a long exact sequence.

2) For each commutative diagram

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

$$0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$$

in $C$ with exact rows,

$$T^i(A'') \xrightarrow{S^i} T^{i+1}(A')$$

$$T^i(B'') \xrightarrow{S^i} T^{i+1}(B')$$

A $D$-functor is universal if any other $D$-functor $T'$ from $C$ to $D$ and natural transformation unique $T^0: T^0 \rightarrow T^0$ there are all natural transformations $\beta^i: T^i \rightarrow T'^i$ making all squares

$$T^i(A'') \rightarrow T^{i+1}(A')$$

$$T^i(A'') \rightarrow T^{i+1}(A')$$
Recall: An injective object $I$ of an abelian category $\mathcal{C}$ is an object for which $\mathbb{H}(-, I)$ is an exact functor.

Thm [Enochs–Esteban 2005] For any scheme $X$, $\text{Qcoh}(X)$ has enough injectives: for every $N \in \text{Qcoh}(X)$ there is an injective object $I$ and a monomorphism $N \rightarrowtail I$.

Thm [Grothendieck] If $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor of abelian categories, $F$ is left exact, and $\mathcal{C}$ has enough injectives, then there is a universal $F$-functor

$$(R^if)_\ast$$

for $F$.

These are the right-derived functors of $F$.

Def: The cohomology of a quasicoherent sheaves on a scheme $X$ is defined by

$$H^i(X, F) = R^i\Gamma(X, (-)^\ast F).$$

Q: How to compute?

Grothendieck shows existence in terms of injective resolutions. Do you know any injective resolutions??
Thm [Grothendieck]: \( H^i(\text{Spec} R, F) = 0 \) \( \forall i > 0 \),
common rings \( R \), Quillen sheaves \( F \) in \( \text{Spec} R \).

pf. See EGA III. 1. Or, [117], III.3 for noetherian case.

Čech Cohomology
let \( \{ U_i \}_{i \in I} \) be an open covering of separated scheme/algebraic variety \( X \), by affine open subnets.

**Fact** For each \( (i_0, \ldots, i_p) \in I^p \),

\[
U_{i_0, \ldots, i_p} = U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p}
\]
is affine.

**Def** For a (quasi-coherent) sheaf \( \mathcal{F} \) on \( X \), the Čech complex with respect to covering \( \mathcal{U} \) is

\[
\check{C}^*(\mathcal{F}) = \bigoplus_{i_0} \prod_{i \in I} \check{H}(U_i, \mathcal{F}) \to \prod_{(i_0, i) \in I^2} \check{H}(U_{i_0} \cap U_{i_1}, \mathcal{F}) \to \ldots
\]

where, for each \( (i_0, \ldots, i_p) \), have, for \( \alpha_{i_0, \ldots, i_p} \in \check{H}(U_{i_0} \cap \ldots \cap U_{i_p}, \mathcal{F}) \),

\[
\delta^p(\alpha)_{(i_0, \ldots, i_p)} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0, \ldots, \hat{i}_k, \ldots, i_p} \cap U_{i_0} \cap \ldots \cap U_{i_k} \cap U_{i_{k+1}} \cap \ldots \cap U_{i_p}
\]

There is another, smaller, Čech complex \( \check{C}^*_{sm}(\mathcal{F}) \):

\[
\prod_{i \in I} \check{H}(U_i) \to \prod_{(i_0, i) \in I^2} \check{H}(U_{i_0} \cap U_{i_1}, \mathcal{F}) \to \ldots
\]

where we've chosen a (strict) ordering on \( I \), and used restricted differentials.

**Prop** \( H^i(\check{C}^*_{sm}(\mathcal{F})) \cong H^i(\check{C}^*(\mathcal{F})) \) for all \( i \geq 0 \).
The advantage of the larger Kottwitz complex is that it is useable more generally, whereas the smaller one, e.g., is a disaster in the étale topology.

Thus $X$ a separated scheme, $\{U_i\}$ an affine open cover, $F \in \mathcal{C}^0_{\text{coh}}(X)$. Then

$$\check{H}^i(U, F) \equiv H^i(X, F) \text{ for all } i \geq 0.$$

Cor If $X$ has an open cover with $n+1$ affine opens, then $H^i(X, F) = 0 \forall i \geq n+1, \forall F \in \mathcal{C}^0_{\text{coh}}(X)$.

P. Use $C^0_{\text{sw}}(F)$.

Ex. $X = \mathbb{P}^1$, affine open cover $U_0, U_1$ by $\mathbb{A}^1$'s.

Recall $\mathcal{O}(n)$ defined by gluing trivial line bundle on the two $\mathbb{A}^1$'s with coords

$s/t, t$, via transition function

$aU_0 \to a^{-1}U_1$, $\varphi_{1,0} : \mathcal{O}_{U_0}|_{U_0 \cap U_1} \to \mathcal{O}_{U_1}|_{U_0 \cap U_1}$

$\varphi_{1,0}(t \frac{a}{s}) = (\frac{t}{s})^n f(a/t)$. 
Thus, let’s take $\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \text{Op}(n)$, get
\[ \mathcal{C}_{\text{sw}}(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \left( k \left[ \frac{s}{t} \right] \oplus k \left[ \frac{t}{s} \right] \right) \xrightarrow{\alpha^0} \bigoplus_{n \in \mathbb{Z}} k \left[ \frac{s}{t}, \frac{t}{s} \right] \]
with $\mathcal{F}^0(f_n, g_n)_{n \in \mathbb{Z}} = (f_n(\frac{s}{t}), g_n(\frac{t}{s}))_{n \in \mathbb{Z}}$.

I claim we get an isomorphism of complexes
\[ \bigoplus_{n \in \mathbb{Z}} k \left[ \frac{s}{t} \right] \oplus k \left[ \frac{t}{s} \right] \xrightarrow{\alpha^0} \bigoplus_{n \in \mathbb{Z}} k \left[ \frac{s}{t}, \frac{t}{s} \right] \]
\[ \downarrow \alpha^1 \]
\[ k[s, t^{-1}] \oplus k[s^{-1}, t] \xrightarrow{\alpha} k[s^{\pm 1}, t^{\pm 1}] \]
by $\alpha^0(f_n(\frac{s}{t}), g_n(\frac{t}{s})) = (f_n(\frac{s}{t})^n, g_n(\frac{t}{s})^n)$,
\[ \alpha^1(h(\frac{s}{t}, \frac{t}{s})) = h(\frac{s}{t}, \frac{t}{s}) \frac{1}{s^n} \]

where $S(F, G) = F - G$.

Now second complex is complex of $k[s, t]$-modules, in fact graded ones.

$\ker(\alpha) = k[s, t]$, clearly?

$\text{coker}(\alpha) = k[s^{\pm 1}, t^{\pm 1}] / k[s, t^{-1}] \otimes k[s^{\pm 1}, t]$ 
\[ = s^{-1}t^{-1}k[s^{-1}, t^{-1}] \text{ but not actually a } k[s^{\pm 1}, t^{-1}] \text{-module!} \]
Do you remember construction of quasi-coherent sheaf $\mathcal{N}$ on $\text{Proj}^n(S)$ from graded $S$-module $M$?

On affine open $\text{Spec } S_d$, $d \geq 0$, 

$\mathcal{N}(\text{Spec } S_d) = (M_d)_0$. [Recall] $S_d = (S_f)_0$.

Claim: These determine a sheaf.

Define $O_{\mathcal{N}}(m) = k[x_0, \ldots, x_n](m)$.

Thus, for $\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} O_{\mathcal{N}}(m)$, "easy" to compute sections on $U_i \cap U_j = U_i \cap U_j$.

$U_{ij} = \text{Spec } k[x_0, \ldots, x_n](x_i)$.

Problem

1. Prove $H^0(\mathcal{N}^n, \mathcal{F}) = k[x_0, \ldots, x_n]$.

2. Prove $H^n(\mathcal{N}^n, \mathcal{F}) = \bigoplus_{i=0}^{n} x_i^{-1} k[x_0, \ldots, x_{i-1}]$ as graded vector spaces.

3. [Hardest] Prove $H^i(\mathcal{N}^n, \mathcal{F}) = 0$ for $0 < i < n$.

Hint: By induction on $n$. $n = 0$ is trivial.

- Use exact sequence $0 \rightarrow (\mathcal{N}^n) \rightarrow k[x_0, \ldots, x_n] \rightarrow \frac{k[x_0, \ldots, x_n]}{(x_i)}$ to get l.e.s. in cohomology.
What is map $H^i(X, F(-1)) \to H^i(X, F)$ given by?

Conclude with by $x_n$ on $H^i(X, F)$ is isomorphism.

OTOH, consider $\mathcal{C}^\infty_0(F)[x_n^{-1}]$.

Claim: It is a Čech complex for Spec $k[x_0, \ldots, x_n]/(x_n)$.
Math 632. Čech Cohomology and Alternating Cochains

Let \( \mathcal{U} = \{ U_i \}_{i \in I} \) be an arbitrary open covering of a topological space \( X \) and let \( \mathcal{F} \) a sheaf of abelian groups on \( X \). This data is fixed. Let \( C^*(\mathcal{F}) \) denote the ‘unrestricted’ Čech complex of \( \mathcal{F} \) with respect to \( \mathcal{U} \), namely (for \( n \geq 0 \))

\[
C^n(\mathcal{F}) = \prod_{i \in I^{n+1}} \Gamma(U_i, \mathcal{F}),
\]

where for \( i = (i_0, \ldots, i_n) \in I^{n+1} \) we define \( U_i = U_{i_0} \cap \cdots \cap U_{i_n} \), and the differential maps are given by the habitual formula. The symmetric group \( S_m \) acts on the usual manner on the set \( I^m = \text{Hom}(\{1, \ldots, m\}, I) \) for each integer \( m \geq 1 \). Using this action, we define \( \mathcal{C}^*_\mathcal{F}(\mathcal{F}) \) to be the subcomplex consisting of alternating Čech cochains (i.e., those cochains \( s_i \in C^m(\mathcal{F}) \) for \( n \geq 0 \) with \( s_{\sigma(i)} = \text{sgn}(\sigma)s_i \) for \( \sigma \in S_{n+1} \) and \( s_i = 0 \) if some \( i_r = i_s \) for \( r \neq s \). Fixing a well-ordering on the set \( I \), we define the quotient complex \( \overline{C}^*(\mathcal{F}) \) via

\[
\overline{C}^n(\mathcal{F}) = \prod_{i_0 < \cdots < i_n} \Gamma(U_i, \mathcal{F})
\]

(for \( n \geq 0 \) and again we use the usual differentiation formulae.

The composite map of complexes \( C^*_\mathcal{F}(\mathcal{F}) \to C^*(\mathcal{F}) \to \overline{C}^*(\mathcal{F}) \) is clearly an isomorphism, so we get an induced diagram on cohomology \( H^*_\mathcal{F}(\mathcal{U}, \mathcal{F}) \to H^*(\mathcal{U}, \mathcal{F}) \to H^*(\overline{C}^*(\mathcal{F})) \) such that the composite map is an isomorphism. We claim that both intermediate maps are isomorphisms (the first map being canonical in the sense that it makes no reference to a choice of well-ordering). In order to prove this, it is enough to analyze the right-hand map, which is visibly surjective, and to prove that this map is an injection. We will prove this by constructing an appropriate homotopy operator.

More precisely, we will construct maps \( K_n : C^n(\mathcal{F}) \to C^{n-1}(\mathcal{F}) \) for each \( n \geq 1 \) such that

\[
K_{n+1} d_n + d_{n-1} K_n = h_n - 1
\]

for all \( n \geq 1 \), where \( h_n : C^n(\mathcal{F}) \to \overline{C}^n(\mathcal{F}) \simeq C^*_\mathcal{F}(\mathcal{F}) \). From this, it follows that the induced composite maps \( H^*(C^*_\mathcal{F}(\mathcal{F})) \to H^*(\overline{C}^*(\mathcal{F})) \simeq H^*(C^*_\mathcal{F}(\mathcal{F})) \to H^*(\overline{C}^*(\mathcal{F})) \) are injective in positive degree. In particular, the left maps are injective in positive degree, which is what we wanted to prove (we take care of degree 0 by observing that all of the degree 0 cohomology groups are identified with the identity map on \( \Gamma(X, \mathcal{F}) \), since \( \mathcal{F} \) is a sheaf). Of course, we can suppose at this time that \( X \) is a non-empty space, so \( I \) is a non-empty set. This will come up shortly.

In order to construct the homotopy operator \( K_\alpha \), we will first consider a more formal ‘dual’ situation which clarifies the role of the index set in our manipulations. For each \( n \geq 0 \), define \( C_n(I) \) to be the free abelian group on the set \( I^{n+1} \), with the usual chain complex structure determined by the formula

\[
\partial(i_0, \ldots, i_n) = \sum_{r=0}^{n} (-1)^r (i_0, \ldots, \hat{i}_r, \ldots, i_n)
\]

(here we are using the canonical injection \( I^{n+1} \to C^n(I) \) of sets for each \( n \geq 0 \) in order to identify the standard ‘basis’ of \( C^n(I) \) with the set \( I^{n+1} \)). This complex is simpler than the Čech complex for a couple of reasons. First of all, many Čech manipulations are purely combinatorial in the index data, so by eliminating the sheaf data which is just ‘carried along’, the situation is made less cluttered. Also, the objects \( C_n(I) \) are direct sums rather than direct products, so it is easier to construct certain maps in this context.

There is a subcomplex \( j : \overline{C}_\alpha(I) \to C_n(I) \) with \( \overline{C}_\alpha(I) \) the free abelian group on those \( (i_0, \ldots, i_n) \in I^{n+1} \) with \( i_0 < \cdots < i_n \) (this is the same well-ordering on \( I \) which was chosen above). We also have a projection map \( p : C_n(I) \to \overline{C}_\alpha(I) \) which annihilates those \( i \in I^{n+1} \) for which some \( i_r = i_s \) with \( r \neq s \), and otherwise sends \( i \) to \( sgn(\sigma)(i_{\sigma(0)}, \ldots, i_{\sigma(n)}) \), where \( \sigma \in S_{n+1} \) is the unique element for which \( i_{\sigma(0)} < \cdots < i_{\sigma(n)} \). Since \( S_{n+1} \) is generated by transpositions of the form \((t, t+1)\) for \( t \in \mathbb{Z}/(n+1) \), it is easy to check that \( p \) is a map of complexes. Clearly \( p\beta = 1 \), though we won’t use this (but note that this is essentially why the above cohomology map \( H^*(\mathcal{U}, \mathcal{F}) \to H^*(\overline{C}^*(\mathcal{F})) \) is surjective; our proof of injectivity will make clear the precise sense in which these two facts are related).
Using the ‘augmentation’ $C_0(I) \to Z$ given by $i \mapsto 1$ for all $i \in I$, we get a complex $C_\bullet(I) \to Z \to 0$. We claim this complex is exact, so it is a free (or more conceptually, projective) resolution of $Z_\bullet$ in the category of abelian groups. Exactness at $Z$ is clear (recall $I$ is not empty!). To handle the higher degree terms, we define a homotopy operator. Pick any $\iota \in I$ ($I$ is non-empty!), and define $\kappa_\iota : C_n(I) \to C_{n+1}(I)$ for each $n \geq 0$ by $\kappa_\iota(t_0, \ldots, t_n) = (t_\iota, t_0, \ldots, t_n)$. Moreover, if we define $C_{-1}(I) = Z$ and define $\partial_0 : C_0(I) \to C_{-1}(I)$ to be the augmentation map, we can even compatibly define $\kappa_i : C_{-1}(I) \to C_0(I)$ by $1 \mapsto i$.

It is easy to check that $\partial_\iota + \kappa_\iota \partial = 1$ on each $C_n(I)$ for $n \geq 0$. Thus, $C_\bullet(I)$ with the augmentation map is indeed an exact complex. Since $j^p : C_\bullet(I) \to C_\bullet(I)$ is a map of complexes respecting the augmentation, it follows from the general theory of projective resolutions that there exist homotopy operators $k_n : C_n(I) \to C_{n+1}(I)$ for each $n \geq 0$ such that $j_n p_n - 1 = \partial_{n+1} k_n + k_{n-1} \partial_n$ for all $n \geq 0$. More precisely, we will prove the existence of maps $k_m : C_m(I) \to C_{m+1}(I)$ ($m \geq 0$) such that $j_n p_m - 1 = \partial_{m+1} k_m + k_{m-1} \partial_m$ for $m \geq 0$ (where $k_{-1} = 0$), with the extra property that for each $i \in I^{n+1}$ ($n \geq 0$), $k_n(i)$ has support inside of the support of $i$ (we define the support of an element $\sum a_i i \in C_n(I)$ for $n \geq 0$ to be the subset of $I$ consisting of the ‘coordinates’ of those $i \in I^{n+1}$ for which $a_i \neq 0$). We describe this by saying “$k_n$ does not increase support”.

We recursively construct the maps. More precisely, suppose for some $n \geq 0$ that we are given maps $k_m : C_m(I) \to C_{m+1}(I)$ for $0 \leq m \leq n$ which do not increase support, and with

$$j_n p_m - 1 = \partial_{m+1} k_m + k_{m-1} \partial_m$$

(we define $k_{-1} = 0$). For example, this is true when $n = 0$ by taking $k_0 = 0$ — note that $j_0 p_0$ is equal to the identity map! Define $k_{n+1}$ by

$$k_{n+1}(i) = \kappa_i(\omega(i)),$$

where $\omega(i) = j_{n+1} p_{n+1}(i) - 1 = k_n \partial_{n+1}(i)$ for $i \in I^{n+2}$. Note that (by the formulas for $\partial$, $j$, $p$, and $\kappa_i$, and our inductive hypothesis on $k_n$) $k_{n+1}(i)$ has support inside of $i$. Also, since $j$ and $p$ are complex maps and $j_n p_n - 1 - \partial_{n+1} k_n = k_{n-1} \partial_n$, we easily compute $\partial_{n+1}(\omega(i)) = k_{n-1} \partial_n \partial_{n+1}(i) = 0$. Thus, $\omega(i) = \partial_{n+1} k_n(\omega(i))$ (note the dependence of $\kappa$ on $i$), and it is straightforward to check that $\partial_{n+2} k_{n+1} + j_{n+1} p_{n+1} - 1 - k_n \partial_{n+1}$ take the same value on each $i \in I^{n+2}$. This completes the construction of the $k_n$'s.

Now we make a functorial observation which will allow us to construct the sought-after homotopy operator in Čech cohomology. Fix $n, m \geq 0$ and a group map $j : C_n(I) \to C_m(I)$ with the property that for each $c \in C_n(I)$, the support of $f(c)$ is a subset of the support of $c$ (equivalently, this can be checked on each $\xi \in I^{n+1}$, and is loosely described by saying that “$f$ doesn’t increase support”). Define a map

$$\bar{f} : C^m(\mathcal{F}) \to C^n(\mathcal{F})$$

by

$$\bar{f}(s_i) = \sum a_{i^\prime} \text{res}^i_{i^\prime} (s_{i^\prime}),$$

where $f(s_i) = \sum a_{i^\prime} s_{i^\prime}$. Here, $i^\prime \in I^{m+1}$ runs through the finitely many elements with support inside of the support of $i$, and

$$\text{res}^i_{i^\prime} : \Gamma(U_{i^\prime}, \mathcal{F}) \to \Gamma(U_i, \mathcal{F})$$

is the usual restriction map (note that $U_i \subset U_{i^\prime}$ because of the the support condition on $i^\prime$).

It is easy to check that $\bar{f}$ is a group map, $id = id$, and if $g : C_n(I) \to C_m(I)$, $h : C_m(I) \to C_r(I)$ are two other such group maps which don’t increase support, then $f + g$ and $h \circ f$ also don’t increase support and $f + g = \bar{f} + \bar{g}$, $h \circ f = \bar{f} \circ \bar{h}$. Since $\partial_{i^\prime} = d_{n-1}$ for each $n \geq 1$ (this is why $\partial_{i^\prime}$ was defined as it was for $n \geq 1$!), it follows that $k_n d_{n} + d_{n-1} k_{n-1} = j_n p_n - 1$ for each $n \geq 1$. But for each $n \geq 0$, $j_n p_n : C_n(\mathcal{F}) \to C^n(\mathcal{F})$ is exactly the map denoted $k_n$ above. Thus, we have the desired homotopy operators $K_n \overset{\text{def}}{=} k_{n-1}$ in each degree $n \geq 1$. 

Last Time \[ S = \mathbb{Z}[x_0, \ldots, x_n], \text{ or indeed} \]
\[ S = \mathbb{R}[x_0, \ldots, x_n], \text{ for any comm ring with 1} \]
\[ M = \bigoplus_{m \in \mathbb{Z}} S(m), \]
\[ f = M \in \text{Qcoh} \left( \text{Proj} S \right) = \text{QCoh} \left( \mathcal{P}_R \right). \]

Form Čech complex \( \check{C}^\ast (\mathcal{F}) \) for open cover by \( U_i = \text{Spec}(S_{x_i}) \).

Observation: \( \check{C}^\ast (\mathcal{F}) \) given by
\[
\begin{array}{cccccc}
\bigoplus_{i=0}^{n} S_{x_i} & \overset{\partial^0}{\longrightarrow} & \bigoplus_{i < j} S_{x_i x_j} & \overset{\partial^1}{\longrightarrow} & \cdots & \overset{\partial^{n-1}}{\longrightarrow} \\
\end{array}
\]

Easy
\[ \ker(\partial^0) = S \text{ as graded } S\text{-modules}. \]
\[ \text{[why is it a graded } S\text{-module?]} \]
\[ \text{coker}(\partial^{n-1}) = S_{x_0 \ldots x_n}/\sum_{i=0}^{n} S_{x_0 \ldots \hat{x_i} \ldots x_n} \]

as graded \( S\text{-modules} \]
\[ = x_0^{-1} \ldots x_n^{-1} \mathbb{R}[x_0^{-1}, \ldots, x_n^{-1}] \] as graded \( \mathbb{R}\text{-modules} \).
Now use \[ 0 \to M(-1) \overset{x_n}{\to} M \to M/x_nM \to 0, \]

get
\[ 0 \to \tilde{C}_{sw}(\mathcal{F}(-1)) \to \tilde{C}_{sw}(\mathcal{F}) \to \tilde{C}_{sw}(\mathcal{F}/x_n\mathcal{F}) \to 0 \]

exact.

Claim \((+) \otimes S/\langle x_n \rangle \) is

\[
\bigoplus_{i=0}^{\infty} S_{x_i}/\langle x_n \rangle \to \bigoplus_{0 < i < n} S_{x_i}(x_n)/\langle x_n \rangle \to \ldots
\]

which equals Čech complex assoc. to \(M/x_nM\) as \(R[x_0, \ldots, x_{n-1}]\)-module.

(not just quasi-isomorphic).

Get
\[
0 \to H^0(P^n_R, \mathcal{F}(-1)) \overset{x_n}{\to} H^0(P^n_R, \mathcal{F}) \to H^0(P^n_R, M/x_nM) \to \ldots
\]

\[
H^1(P^n_R, \mathcal{F}(-1)) \overset{x_n}{\to} H^1(P^n_R, \mathcal{F}) \to H^1(P^n_R, M/x_nM) \to \ldots
\]

\[
\to H^n(P^n_R, \mathcal{F}(-1)) \to H^n(P^n_R, \mathcal{F}) \to H^n(P^n_R, M/x_nM) \to H^n(P^n_R, M) \to 0
\]
Prove by induction on $n$.

\[(H) \quad H^i(C_k, \xi^{i+1}) \cong \text{hom}_{n-k} H^i(M_k, \eta) \text{ is an isomorphism for } k \geq 0, 0 < i < n.\]

\[n = 1 \text{ case clear!}\]

\[\text{Observe on } H^0, \text{ get}\]

\[H^0(\mathbb{P}^n_k, \xi) \rightarrow H^0(\mathbb{P}^n_k, \frac{\mathbb{X}}{\mathbb{X} \times M})\]

\[S \rightarrow S/\mathbb{X} \text{ surjective,}\]

\[\text{so } H^1(\mathbb{P}^n_k, \xi^{i+1}) \rightarrow H^1(\mathbb{P}^n_k, \eta) \text{ injective.}\]

By inductive hypothesis, conclude \((H)\) isomorphism for \(0 < i < n-2\). Have also

\[H^{n-2}(\mathbb{P}^n_k, \frac{\mathbb{X}}{\mathbb{X} \times M}) \rightarrow H^{n-1}(\mathbb{P}^n_k, \xi^{i+1}) \rightarrow H^n(\mathbb{P}^n_k, \eta)\]

\[\rightarrow H^n(\mathbb{P}^n_k, \frac{\mathbb{X}}{\mathbb{X} \times M}) \rightarrow H^n(\mathbb{P}^n_k, \xi^{i+1}).\]

What is last map? It's boundary map, coming from snake lemma. Calculate using...
Given \( f \in S_{x_0\ldots x_{n-1}}/\langle x_n \rangle \), lift to \( \tilde{f} \in S_{x_0\ldots x_{n-1}} \). Now \( \varphi^{-1}(\tilde{f}) = \pm \tilde{\varphi} \), which can be written \( \varphi^{-1}(\tilde{f}) = \pm \tilde{\varphi} = \pm x_n \cdot (x_n \tilde{f}) \).

Now project \( \pm x_n \tilde{f} \) into \( S_{x_0\ldots x_n}/\langle x_n \rangle \). Let:

\[
H^\ast(\mathbb{P}^n, \mathbb{R}, X) = \bigoplus_i \mathbb{Z} S_{x_0\ldots x_{n-1}}/\langle x_n \rangle
\]

That is image of \( f \) under corresponding homomorphism. When does it go to zero? Exactly if it's polynomial in some variable, say \( x_i \). Then \( \tilde{f} \) was also polynomial in \( x_i \), so:

\[
\text{Image of } f \in H^\ast(\mathbb{P}^n, \mathbb{R}, X) / \bigoplus_i \mathbb{Z} S_{x_0\ldots x_{n-1}}/\langle x_n \rangle
\]
was zero already. We conclude that
\[ H^n(M^{n-1}, \mathbb{R}, M^M) \to H^n(M^R, F) \]
is injective, so
\[ H^n(M^R, F) \cong H^n(M^R, F) \]
is surjective too.

Thus
\[ H^i(M^R, F) \cong H^i(M^R, F) \]
is an isomorphism for \( 0 < i < n \).

New reason \( C^\infty(V) \) with \( S_{xn} \). By "Thus",
\[ H^i(C^\infty(V)) \otimes S_{xn} \cong H^i(C^\infty(V)) \text{ for } 0 < i < n. \]

But also, \( C^\infty(V) \otimes S_{xn} \cong C^\infty(V, F_{xn}) \),
where \( V \) is open cover of \( U_n \) by
open sets \( V_i = U_i \cap U_n \). So
\[ H^i(C^\infty(V)) = 0 \text{ for } i > 0. \]

Thus \[ H^i(M^R, F) = 0 \text{ for } 0 < i < n. \]
This proves inductive step.
Tangent Sheaf \( X \) a smooth variety over \( k \).

\( U = \text{Spec} \, R \subseteq X \) affine open.

**Def** \( T_X(U) = \text{Der}_K(R, R) = \{ d: R \to R \mid d(\lambda r_z) = \lambda d(r_z) + d(\lambda) r_z \} \)

sheaf of derivations or tangent sheaf of \( X \).

**Facts**

1. It is dual to sheaf \( \Omega^1_X \) of Kähler differentials, \( T_X = \text{Hom}_X(\Omega^1_X, \Omega^1_X) \).

   [Do you know what it means?]

2. If \( R = k[x_1, \ldots, x_n] \),

   \( T_{\text{Spec} \, R} = \text{Der}_K(R, R) = \bigoplus_{i=1}^n k[x_1, \ldots, x_n] \frac{\partial}{\partial x_i} \).

3. If \( X \to Y \) is a morphism of smooth varieties, we get a homomorphism \( T_X \to f^* T_Y \) of coherent \( O_X \)-modules.

This is usual calculus.
COS, Day 8

Last Time: $B \subseteq \text{Top}(X)$ a basis of the topology on $X$.

A sheaf on $B$ is a functor $B^P \to \text{Ab}$

so that after all $U \in B$ and coverings $U_i \in B$ of $U$

$\phi(U) \to \prod_i \phi(U_i)$ is injective,

and (2) for all coverings $\mathcal{U}^{U_k}_{ij}$ of each $U_i \cap U_j$,

$0 \to \phi(U) \to \prod_i \phi(U_i) \to \prod_{i,j} \phi(U_{ij})$

is exact.

Prop: "Restriction"

$\text{Shv}(X) \to \text{Shv}(B)$

is exactly an equivalence of categories.

Tangent sheaf on X: defined on affine open $\text{Spec } R \times X$

by $T_x(\text{Spec } R) = \text{Der}_k (R, R)$.

Kähler differentials

$\Omega_{A/k} = \mathbb{K}$ free $A$-module on symbols $dr, re^R$,
modulo submodule generated by elements
\[ d(r+r') - dr - dr', \]
\[ d(s), \quad r, r', R, \]
\[ d(\langle r \rangle) - \phi(r) - r' \phi(w) \quad \text{see k}. \]

**Universal Property**
Every \( k \)-linear derivation of \( R \)-modules, \( R \otimes k \), factors uniquely through \( R \otimes k \xrightarrow{d} \Omega^1_R \).

Thus \( \text{Der} (R, R) = \text{Hom}_k \left( \Omega^1_R, R \right) \).

So \( T_X = \text{hom}_{\Omega^1} \left( \Omega^1_X, \Omega^1_X \right) \).

\( \Omega^1_X \) is a sheaf of \( \Omega_X \)-modules. How to show that?

1. Define \( I_\Delta = \ker \left( \Omega_X \xrightarrow{res} \Omega^1_X (x) \right) \).
2. Define \( \Omega^1_X / \Omega^1_X \) with \( \Delta : \Omega_X \rightarrow I_{\Delta} / I_{\Delta} \).

Check \( \Omega^1_X (\text{Spec} R) \) compatibly with \( d : \Omega_X \rightarrow I_{\Delta} / I_{\Delta} \).

\[ d(f) = \phi(f) f^2 \mod \Delta. \]

Implies \( \text{Der}_k (R, R) \) gives sheaf on \( \Omega \)-basis, hence defines sheaf.

See Hartshorne or Vakil for details.
Crucial Computation If \( R = k[x_1, \ldots, x_n] \), then
\[
D_{x_i}^! = \bigoplus_{i=1}^n Rdx_i.
\]

Then \( \text{Der}_k(R, R) = \bigoplus_{i=1}^n R \frac{\partial}{\partial x_i} \),

where \( \frac{\partial}{\partial x_i} \) is usual partial derivative.

Fact Given morphism \( f: X \to Y \), have
natural map \( T_X \to f^*T_Y \).

Why On affine, say \( \text{Spec} R \to \text{Spec} S \), \( S \to R \)
want \( \text{Der}_k(R, R) \to R \otimes \text{Der}_k(S, S) \).

Note that \( R \)-module structure on \( \text{Der}_k(S, S) \)
is given by \text{mult.} on output.

Using that \( \text{Spec} S \) is smooth, we get an
isomorphism
\[
R \otimes \text{Der}_k(S, S) \cong R \otimes \text{Hom}_S(S_k, S)
\cong \text{Hom}_R(R \otimes S_k, R)
\cong \text{Der}_k(S, R). \text{ Thus map is given by}
\text{Der}_k(R, R) \to \text{Der}_k(S, R)
\phi \mapsto \phi \circ f^#.
\]
Example Consider $A^{n+1} \times 0 \rightarrow P^n$

$v_i \mapsto (v_1^i)$.

Over affine open $U_i = \text{Spec } k[x_0, \ldots, x_n] \subseteq P^n$,

get $\emptyset \mapsto f^{-1}(U_i) \rightarrow U_i$

$A^{n+1}

(x_0, \ldots, x_n) \mapsto (\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}) = (u_0, \ldots, u_i, \ldots, u_n)

\text{"dual" for}

k[x_0, x_1^{+}, \ldots, x_n] \hookrightarrow \bigoplus_{u_j} k[u_0, \ldots, u_n]

\frac{x_i}{x_i} \mapsto u_j$

Expect a map

\bigoplus_{l=1}^{n+1} k[x_0, x_1^{+}, \ldots, x_n] \frac{\partial}{\partial x_l} \rightarrow \bigoplus_{l \neq i} k[x_0, x_1^{+}, \ldots, x_n] \frac{\partial}{\partial u_j}

\frac{\partial}{\partial x_l} (f(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i})) = \sum_{j} \frac{\partial f}{\partial u_j} \frac{\partial u_j}{\partial x_l}

\# Now \frac{\partial u_j}{\partial x_l} = \begin{cases} \frac{x_j}{x_i^2}, & l = i \\ \frac{1}{x_i}, & l = j \neq i \\ 0, & \text{otherwise} \end{cases}

So \frac{\partial}{\partial x_l} \mapsto \frac{1}{x_i} \frac{\partial u_l}{\partial x_l}, \ l \neq i

\frac{\partial}{\partial x_i} \mapsto -\sum_{j} \frac{x_j}{x_i^2} \frac{\partial u_j}{\partial x_i}.
These images do not "descend" to $U_i$: they are not pullbacks of vector fields on $U_i$. What do descend are

$$X_\ell \frac{\partial}{\partial x_\ell} \mapsto u_\ell \frac{\partial}{\partial u_\ell}, \quad \ell \neq i,$$

$$X_i \frac{\partial}{\partial x_i} \mapsto -\sum_j y_j \frac{\partial}{\partial u_j}.$$  

But it's now clear each of these gives a section of $\Omega^1$ that vanishes along a hyperplane in $\mathbb{P}^n$. We thus get

$$\mathcal{O}(1)^{n+1}\bigg|_{\mathbb{P}^n} \to \mathcal{O}(1)^{n+1}.$$  

Claim: It is surjective.  

[Why? See local computation]

Claim: Its kernel is spanned by $\sum_{\ell=0}^n X_\ell \frac{\partial}{\partial x_\ell}.$

Thus

One gets an exact sequence on $\mathbb{P}^n$,

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}(1)^{n+1} \to \mathcal{H}^1(\mathcal{O}_{\mathbb{P}^n}) \to 0.$$  

Dual sequence is

$$0 \to \mathcal{O}(1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{H}^1(\mathcal{O}_{\mathbb{P}^n}) \to 0.$$  

"Euler sequence."