Math 595 Cohomology of Schemes, Day 7 Problems

Consider the formulas for vector fields X_i on $\mathbb{A}_k^{n+1}$ ($n \geq 1$ throughout, k a commutative ring with 1) defined by $X_i = x_i \frac{\partial}{\partial x_i}$.

Problem 1. Show that each of these vector fields is invariant under the scaling action of the multiplicative group scheme $\mathbb{G}_m = \mathrm{Spec}(k[t^{\pm 1}])$ on $\mathbb{A}^{n+1} \setminus \{0\}$, and conclude that each defines a vector field on $\mathbb{P}^n$.

Problem 2. Show by a calculation in local coordinates that each of $X_0, \ldots, X_n$ vanishes along a hyperplane. Hence, each actually defines a sheaf map $\mathcal{O}(1) \xrightarrow{X_i} T_{\mathbb{P}^n}$, where $T_{\mathbb{P}^n}$ is the tangent sheaf of $\mathbb{P}^n$.

Problem 3. Show that these (twisted) sections generate $T_{\mathbb{P}^n}$, in the sense that the natural map $\phi : \mathcal{O}(1)_{\mathbb{P}^n}^{n+1} \xrightarrow{(X_0,\ldots,X_n)} T_{\mathbb{P}^n}$ is surjective. [Hint: the kernel is the infinitesimal generator of the scaling action.]

Problem 4. Prove that the kernel of ϕ from the previous item is isomorphic to $\mathcal{O}_{\mathbb{P}^n}$. Deduce, dually, that there is an exact sequence of coherent sheaves on $\mathbb{P}^n$,

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

Conclude that $H^0(\mathbb{P}^n, \Omega^1) = 0$ and $H^1(\mathbb{P}^n, \Omega^1) \cong k$, and $H^i(\mathbb{P}^n, \Omega^1) = 0$ for $i > 1$.

Problem 5. Use standard properties of $\bigwedge^k$ applied to exact sequences of vector spaces to conclude that there is an exact sequence of coherent sheaves on $\mathbb{P}^n$,

$$0 \to \bigwedge^k \Omega^1 \to \bigwedge^k \mathcal{O}(-1)^{n+1} \to \bigwedge^{k-1} \Omega^1 \to 0,$$

that is,

$$0 \to \Omega^k \to \mathcal{O}(-k)^{\binom{n+1}{k}} \to \Omega^{k-1} \to 0.$$

Problem 6. Prove directly that $H^0(\mathbb{P}^n, \Omega^k) = 0$ for $k \geq 1$. Use induction to conclude $\dim H^j(\mathbb{P}^n, \Omega^i) = \delta_{i,j}$ for $0 \leq i \leq n$.