Consider the formulas for vector fields $X_i$ on $\mathbb{A}^{n+1}_k$ ($n \geq 1$ throughout, $k$ a commutative ring with 1) defined by $X_i = x_i \frac{\partial}{\partial x_i}$.

**Problem 1.** Show that each of these vector fields is invariant under the scaling action of the multiplicative group scheme $G_m = \text{Spec}(k[t^{\pm 1}])$ on $\mathbb{A}^{n+1}_k \setminus \{0\}$, and conclude that each defines a vector field on $\mathbb{P}^n$.

**Problem 2.** Show by a calculation in local coordinates that each of $X_0, \ldots, X_n$ vanishes along a hyperplane. Hence, each actually defines a sheaf map $\mathcal{O}(1) \xrightarrow{X_i} \mathcal{T}_{\mathbb{P}^n}$, where $\mathcal{T}_{\mathbb{P}^n}$ is the tangent sheaf of $\mathbb{P}^n$.

**Problem 3.** Show that these (twisted) sections generate $\mathcal{T}_{\mathbb{P}^n}$, in the sense that the natural map $\phi : \mathcal{O}(1)^{n+1}_{\mathbb{P}^n} \xrightarrow{X_0, \ldots, X_n} \mathcal{T}_{\mathbb{P}^n}$ is surjective. [Hint: the kernel is the infinitesimal generator of the scaling action.]

**Problem 4.** Prove that the kernel of $\phi$ from the previous item is isomorphic to $\mathcal{O}_{\mathbb{P}^n}$.

Deduce, dually, that there is an exact sequence of coherent sheaves on $\mathbb{P}^n$,

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$  

Conclude that $H^0(\mathbb{P}^n, \Omega^1) = 0$ and $H^1(\mathbb{P}^n, \Omega^1) \cong k$, and $H^i(\mathbb{P}^n, \Omega^1) = 0$ for $i > 1$.

**Problem 5.** Use standard properties of $\bigwedge^k$ applied to exact sequences of vector spaces to conclude that there is an exact sequence of coherent sheaves on $\mathbb{P}^n$,

$$0 \to \bigwedge^k \Omega^1 \to \bigwedge^k \mathcal{O}(-1)^{n+1} \to \bigwedge^{k-1} \Omega^1 \to 0,$$

that is,

$$0 \to \Omega^k \to \mathcal{O}(-k)^{(n+1)} \to \Omega^{k-1} \to 0.$$

**Problem 6.** Prove directly that $H^0(\mathbb{P}^n, \Omega^k) = 0$ for $k \geq 1$. Use induction to conclude $\dim H^i(\mathbb{P}^n, \Omega^1) = \delta_{i,j}$ for $0 \leq i \leq n$. 

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