Math 595 Cohomology of Schemes, Day 6 Problems

Suppose that S is a nonnegatively graded ring generated in degree 1. [Though this latter assumption is not so important.] Recall that $\mathrm{Proj}(S)$ is covered by affine open subschemes of the form $\mathrm{Spec}(S_{(f)})$, where $S_{(f)}$, for $f \in S_d$, denotes the degree zero part of the (graded) localized ring, $(S_f)_0$.

If M is a graded S-module, the associated quasicoherent sheaf on $\mathrm{Proj}(S)$, written $\widetilde{M}$, has

$$\widetilde{M}\big(\mathrm{Spec}(S_{(f)})\big) = (M_f)_0.$$

Let $S = \mathbb{Z}[x_0, \ldots, x_n]$. Consider the open covering of $\mathbb{P}^n = \mathrm{Proj}(S)$ by affine open sets $U_i = \mathrm{Spec}(S_{(x_i)})$.

Problem 1. Let $M = \bigoplus_{m\in\mathbb{Z}} S(m)$, where $S(m)$ means S with the grading shift for which $S(m)_k = S_{m+k}$. Recall that we write $\mathcal{O}_{\mathbb{P}^n}(m) = \widetilde{S(m)}$. Compute the small Čech complex of M in terms of localizations of M. Show that the small Čech complex of M is naturally a complex of graded S-modules.

Problem 2. Show that $H^0(\mathbb{P}^n, \widetilde{M}) \cong S$ as graded vector spaces.

Problem 3. Show $H^n(\mathbb{P}^n, \widetilde{M}) \cong S[x_0^{-1}, x_1^{-1}, \ldots, x_n^{-1}]/\sum_i S[x_0^{-1}, \ldots, x_{i-1}^{-1}, x_i, x_{i+1}^{-1}, \ldots, x_n^{-1}]$ as graded S-modules.

We now prove by induction on n that $H^i\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)\big) = 0$ for $0 < i < n$. For $n = 1$ it is trivial.

Now, consider the short exact sequence of sheaves on $\mathbb{P}^n$ associated to the exact sequence of S-modules,

$$0 \to S(m-1) \xrightarrow{x_n \cdot -} S(m) \to \big(S/(x_n)\big)(m) \to 0.$$

Problem 4. Show that

$$H^i(\mathbb{P}^n, \widetilde{(S/(x_n))(m)}) \cong H^i\big(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(m)\big).$$

Use this and the inductive hypothesis to conclude that multiplication by x_n on $H^i(\mathbb{P}^n, \widetilde{M})$ is an automorphism for $1 < i < n-1$. Use a more sophisticated long exact sequence argument to show that it is true for $i = 1, n-1$ as well.

Problem 5. On the other hand, prove that tensoring $\check{C}^\bullet_{\mathrm{sm}}(\widetilde{M})$ with S_{x_n} (i.e., localizing the Čech complex) yields the Čech complex of $\widetilde{M}|_{U_n}$. Conclude that $H^i(\mathbb{P}^n, \widetilde{M}) = H^i(U_n, \widetilde{M}|_{U_n}) = 0$ for $0 < i < n$.